

THE E-NORMAL STRUCTURE OF ODD-DIMENSIONAL UNITARY GROUPS

RAIMUND PREUSSER

ABSTRACT. In this paper we define odd-dimensional unitary groups $U_{2n+1}(R, \Delta)$. These groups contain as special cases the odd-dimensional general linear groups $GL_{2n+1}(R)$ where R is any ring, the odd-dimensional orthogonal and symplectic groups $O_{2n+1}(R)$ and $Sp_{2n+1}(R)$ where R is any commutative ring and further Bak's even-dimensional unitary groups $U_{2n}(R, \Lambda)$ where (R, Λ) is any form ring. We classify the E-normal subgroups of the groups $U_{2n+1}(R, \Delta)$ (i.e. the subgroups which are normalized by the elementary subgroup $EU_{2n+1}(R, \Delta)$), under the condition that R is a quasifinite ring with involution and $n \geq 3$. Further we investigate the action of $U_{2n+1}(R, \Delta)$ on the set of all E-normal subgroups by conjugation.

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1. INTRODUCTION

Towards the end of the 19th century mathematicians (e.g. C. Jordan and L. E. Dickson) became interested in the normal subgroups of classical groups over finite fields. Their results were generalised to classical groups over arbitrary fields and skew fields by J. Dieudonne and others in the 1940's and 1950's. In the early 60's, W. Klingenberg generalised the results first to classical groups over commutative local rings and then to classical groups over commutative semilocal rings. In 1964, H. Bass described the E-normal subgroups (i.e. the subgroups normalized by the elementary subgroup) of general linear groups over rings satisfying a so called stable range condition. Namely he proved that if H is a subgroup of the general linear group $GL_n(R)$, then

$$H \text{ is E-normal} \Leftrightarrow \exists! \text{ ideal } I : E_n(R, I) \subseteq H \subseteq C_n(R, I) \quad (1.1)$$

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for all rings R of finite stable range r and $n \geq \max(r+1, 3)$. In (1.1), $E_n(R, I)$ denotes the relative elementary subgroup of level I and $C_n(R, I)$ denotes the full congruence subgroup of level I . In the 70's, I. Golubchik and A. Suslin proved results which imply that (1.1) is true if R is commutative and $n \geq 3$, without a stable range restriction. In 1981, L. N. Vaserstein proved that (1.1) holds if R is almost commutative (i.e. finitely generated as module over its center) and $n \geq 3$.

A natural question is if similar results hold for other classical groups. In 1969, A. Bak proved that if H is a subgroup of the (even-dimensional) hyperbolic unitary group $U_{2n}(R, \Lambda)$, then

$$H \text{ is E-normal} \Leftrightarrow \exists! \text{ form ideal } (I, \Gamma) : EU_{2n}((R, \Lambda), (I, \Gamma)) \subseteq H \subseteq CU_{2n}((R, \Lambda), (I, \Gamma)) \quad (1.2)$$

for all form rings (R, Λ) such that R has finite Bass-Serre dimension d and $n \geq \max(d+2, 3)$. In (1.2), $EU_{2n}((R, \Lambda), (I, \Gamma))$ denotes the relative elementary subgroup of level (I, Γ) and $CU_{2n}((R, \Lambda), (I, \Gamma))$ denotes the full congruence subgroup of level (I, Γ) . The hyperbolic unitary groups (which were first defined in Bak's thesis) include as special cases the groups $GL_{2n}(R)$ where R is any ring and the groups $O_{2n}(R)$ and $Sp_{2n}(R)$ where R is any commutative ring. In 2013, H. You and X. Zhou proved that (1.2) holds if R is commutative and $n \geq 3$ and in 2014, the author proved in his thesis that (1.2) holds if R is almost commutative and $n \geq 3$.

In this paper we define odd-dimensional unitary groups $U_{2n+1}(R, \Delta)$. These groups are precisely Petrov's odd hyperbolic unitary groups such that $V_0 = R$ (note that Petrov's definition is different but equivalent). They contain as special cases the groups $GL_{2n+1}(R)$ where R is any ring, $O_{2n+1}(R)$ and $Sp_{2n+1}(R)$ where R is any commutative ring and further all even-dimensional unitary groups $U_{2n}(R, \Lambda)$ (cf. example 18). We prove that if H is a subgroup of an odd-dimensional unitary group $U_{2n+1}(R, \Delta)$, then

$$H \text{ is E-normal} \Leftrightarrow \exists! \text{ odd form ideal } (I, \Omega) : EU_{2n+1}((R, \Delta), (I, \Omega)) \subseteq H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega)) \quad (1.3)$$

provided R is semilocal or quasifinite (i.e. a direct limit of almost commutative rings) and $n \geq 3$. In (1.3), $EU_{2n+1}((R, \Delta), (I, \Omega))$ denotes the relative elementary subgroup of level (I, Ω) and $CU_{2n+1}((R, \Delta), (I, \Omega))$ denotes the full congruence subgroup of level (I, Ω) . Further we investigate the action of $U_{2n+1}(R, \Delta)$ on the set of all E-normal subgroups by conjugation.

The author expects that the results of this paper can be generalized to all of Petrov's odd unitary groups (over quasifinite rings and such that the Witt index of the underlying quadratic space is at least 3), but the details will probably be even more complicated than in this note.

The paper is organised as follows. In Section 2 we recall some standard notation which will be used throughout the paper. In Section 3 we define the odd-dimensional unitary groups and some important subgroups. In Section 4 we prove the main result (1.3), first for semilocal rings, then for certain Noetherian rings and then for quasifinite rings. In the last section we investigate the action of conjugation on E-normal subgroups, the main result of this section is Theorem 63.

2. NOTATION

Let G be a group and H, K be subsets of G . The subgroup of G generated by H is denoted by $\langle H \rangle$. If $g, h \in G$, let ${}^h g := hgh^{-1}$, $g^h := h^{-1}gh$ and $[g, h] := ghg^{-1}h^{-1}$. Set ${}^K H := \langle \{{}^k h \mid h \in H, k \in K\} \rangle$ and $H^K := \langle \{h^k \mid h \in H, k \in K\} \rangle$. Analogously define $[H, K]$ and HK . Instead of ${}^K \{g\}$ we write ${}^K g$ (analogously we write g^K instead of $\{g\}^K$, ${}^g H$ instead of $\{g\}H$, $[g, K]$ instead of $[\{g\}, K]$ etc.).

In this paper, $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of all natural numbers. The word "ring" will always mean associative ring with $1 \neq 0$, "ideal" will mean two-sided ideal. If R is a ring and $m, n \in \mathbb{N}$, then the set of all invertible elements in R is denoted by R^* and the set of all $m \times n$ matrices with entries in R is denoted by $M_{m \times n}(R)$. If $a \in M_{m \times n}(R)$, let $a_{ij} \in R$ denote the element in the (i, j) -th position. Let $a^t \in M_{n \times m}(R)$ denote its transpose, thus $(a^t)_{ij} = a_{ji}$. Denote the i -th row of a by a_{i*} and the j -th column of a by a_{*j} . We set $M_n(R) := M_{n \times n}(R)$. The identity matrix in $M_n(R)$ is denoted by e or $e^{n \times n}$ and the matrix with a 1 at position (i, j) and zeros elsewhere is denoted by e^{ij} . If $a \in M_n(R)$ is invertible, the entry of a^{-1}

at position (i, j) is denoted by a'_{ij} , the i -th row of a^{-1} by a'_{i*} and the j -th column of a^{-1} by a'_{*j} . Further we denote by nR the set of all rows $v = (v_1, \dots, v_n)$ with entries in R and by R^n the set of all columns $u = (u_1, \dots, u_n)^t$ with entries in R .

3. ODD-DIMENSIONAL UNITARY GROUPS

3.1. Odd form rings and odd form ideals. First we recall the definition of a ring with involution with symmetry and introduce the notion of an odd quadruple.

Definition 1. Let R be a ring and

$$\begin{aligned} \bar{\cdot} : R &\rightarrow R \\ x &\mapsto \bar{x} \end{aligned}$$

an anti-isomorphism of R (i.e. $\bar{\cdot}$ is bijective, $\overline{x+y} = \bar{x} + \bar{y}$, $\overline{xy} = \bar{y}\bar{x}$ for any $x, y \in R$ and $\bar{1} = 1$). Further let $\lambda \in R$ such that $\bar{x} = \lambda x \lambda$ for any $x \in R$. Then λ is called a *symmetry* for $\bar{\cdot}$, the pair $(\bar{\cdot}, \lambda)$ an *involution with symmetry* and the triple $(R, \bar{\cdot}, \lambda)$ a *ring with involution with symmetry*. A subset $A \subseteq R$ is called *involution invariant* iff $\bar{x} \in A$ for any $x \in A$. We call a quadruple $(R, \bar{\cdot}, \lambda, \mu)$ where $(R, \bar{\cdot}, \lambda)$ is a ring with involution with symmetry and $\mu \in R$ has the property that $\mu = \bar{\mu}\lambda$ an *odd quadruple*.

Remark 2. Let $(R, \bar{\cdot}, \lambda, \mu)$ be an odd quadruple.

- (a) It is easy to show that $\bar{\bar{\lambda}} = \lambda^{-1}$.
- (b) If $\lambda \in \text{Center}(R)$ then one gets the usual concept of an involution which is for instance used in [9].
- (c) The map

$$\begin{aligned} \underline{\cdot} : R &\rightarrow R \\ x &\mapsto \underline{x} := \bar{\lambda} \bar{x} \lambda \end{aligned}$$

is the inverse map of $\bar{\cdot}$. One checks easily that $(R, \underline{\cdot}, \lambda, \mu)$ is an odd quadruple.

- (d) For any $n \in \mathbb{N}$, $(M_n(R), *, \lambda e, \mu e)$ is an odd quadruple where $\sigma^* := \bar{\sigma}^t = ((\bar{\sigma}_{ij})_{ij})^t = (\bar{\sigma}_{ji})_{ij}$ for any $\sigma \in M_n(R)$. The inverse map of $*$ is the map $\underline{\cdot}$ which associates to each $\sigma \in M_n(R)$ the matrix $\sigma_* := \underline{\sigma}^t$. It follows from (c) that $(M_n(R), \underline{\cdot}, \lambda e, \mu e)$ is also an odd quadruple.

Below we introduce the notion of an R -quasimodule.

Definition 3. Let R be a ring, $(G, \dot{+})$ a group and

$$\begin{aligned} \bullet : G \times R &\rightarrow G \\ (a, x) &\mapsto a \bullet x := \bullet(a, x) \end{aligned}$$

be a map such that

- (1) $a \bullet 0 = 0$ for any $a \in G$,
- (2) $a \bullet 1 = a$ for any $a \in G$,
- (3) $(a \bullet x) \bullet y = a \bullet (xy)$ for any $a \in G$ and $x, y \in R$ and
- (4) $(a \dot{+} b) \bullet x = (a \bullet x) \dot{+} (b \bullet x)$ for any $a, b \in G$ and $x \in R$.

Then $(G, \dot{+}, \bullet)$ is called a (*right*) R -quasimodule. Let $(G, \dot{+}, \bullet)$ and $(G', \dot{+}', \bullet')$ be R -quasimodules. Then a map $f : G \rightarrow G'$ satisfying $f(a \dot{+} b) = f(a) \dot{+}' f(b)$ for any $a, b \in G$ and $f(a \bullet x) = f(a) \bullet' x$ for any $a \in G$ and $x \in R$ is called a R -quasimodule homomorphism. A subgroup H of G which is \bullet -stable (i.e. $a \bullet x \in H$ for any $a \in H$ and $x \in R$) is called a R -subquasimodule. Further, if $A \subseteq G$ and $B \subseteq R$, we denote by $A \bullet B$ the subgroup of G generated by $\{a \bullet b | a \in A, b \in B\}$.

Remark 4. If $(G, \dot{+}, \bullet)$ is a R -quasimodule, we treat \bullet as an operator with higher priority than $\dot{+}$ (e.g. $a \dot{+} b \bullet x = a \dot{+} (b \bullet x)$).

Next we define an R -quasimodule structure on R^2 . The definition is motivated by the relations of elementary matrices (see Lemma 23, (E1) and (SE2)).

Definition 5. Let $(R, -, \lambda, \mu)$ be an odd quadruple. Define the maps

$$\begin{aligned} \dot{+} : R^2 \times R^2 &\rightarrow R^2 \\ ((x_1, y_1), (x_2, y_2)) &\mapsto (x_1, y_1) \dot{+} (x_2, y_2) := (x_1 + x_2, y_1 + y_2 - \bar{x}_1 \mu x_2) \end{aligned}$$

and

$$\begin{aligned} \bullet : R^2 \times R &\rightarrow R^2 \\ ((x, y), a) &\mapsto (x, y) \bullet a := (xa, \bar{a}ya) \end{aligned}$$

Then $(R^2, \dot{+}, \bullet)$ is an R -quasimodule. We call it *the Heisenberg quasimodule* and denote it by \mathfrak{H} .

Remark 6.

- (a) We denote the inverse of an element $(x, y) \in R^2$ with respect to $\dot{+}$ by $\dot{-}(x, y)$.
- (b) Assume we have fixed an odd quadruple $(R, -, \lambda, \mu)$. $(R, -, \lambda, \underline{\mu})$ defines also an R -quasimodule structure on R^2 . We denote the addition (resp. scalar multiplication) defined by $(R, -, \lambda, \underline{\mu})$ by $\dot{+}_{-1}$ (resp. \bullet_{-1}). We sometimes denote the addition (resp. scalar multiplication) defined by $(R, -, \lambda, \mu)$ by $\dot{+}_1$ (resp. \bullet_1).
- (c) Define the map

$$\begin{aligned} \bullet' : R \times R &\rightarrow R^2 \\ (x, a) &\mapsto x \bullet' a := \bar{a}xa. \end{aligned}$$

Then $(R, +, \bullet')$ is an R -quasimodule.

The following lemma is straightforward to check.

Lemma 7. Let $(R, -, \lambda, \mu)$ be an odd quadruple. Then

- (1) $\dot{-}(x, y) = (-x, -y - \bar{x}\mu x)$ for any $(x, y) \in R^2$,
- (2) $(x_1, y_1) \dot{-} (x_2, y_2) = (x_1 - x_2, y_1 - y_2 + \overline{x_1 - x_2} \mu x_2)$ for any $(x_1, y_1), (x_2, y_2) \in R^2$,
- (3) $[(x_1, y_1), (x_2, y_2)] = (0, \bar{x}_2 \mu x_1 - \bar{x}_1 \mu x_2)$ and
- (4) $\sum_{1 \leq i \leq n} (x_i, y_i) = (\sum_{i=1}^n x_i, \sum_{i=1}^n y_i - \sum_{\substack{i,j=1, \\ i < j}}^n \bar{x}_i \mu x_j)$ for any $n \in \mathbb{N}$ and $(x_1, y_1), \dots, (x_n, y_n) \in R^2$.

The map tr defined below will be used in the definition of an odd form ring.

Definition 8. Let $(R, -, \lambda, \mu)$ be an odd quadruple. Define the map

$$\begin{aligned} tr : R^2 &\rightarrow R \\ (x, y) &\mapsto \bar{x}\mu x + y + \bar{y}\lambda. \end{aligned}$$

One checks easily that $tr : \mathfrak{H} \rightarrow (R, +, \bullet')$ is an R -quasimodule homomorphism.

Next we define an odd form ring.

Definition 9. Let $(R, -, \lambda, \mu)$ be an odd quadruple. Set

$$\Delta_{min} := \{(0, x - \bar{x}\lambda) | x \in R\}$$

and

$$\Delta_{max} := \ker(tr).$$

An R -subquasimodule Δ of \mathfrak{H} lying between Δ_{min} and Δ_{max} is called an *odd form parameter* (for $(R, -, \lambda, \mu)$). Since Δ_{min} and Δ_{max} are R -subquasimodules of \mathfrak{H} , they are respectively the smallest and the largest odd form parameter. If Δ is an odd form parameter for R , the pair $((R, -, \lambda, \mu), \Delta)$ is called an *odd form ring*.

Remark 10. Let Δ be an odd form parameter for $(R, \bar{\cdot}, \lambda, \mu)$.

- (a) Instead of $((R, \bar{\cdot}, \lambda, \mu), \Delta)$ we often write (R, Δ) .
- (b) $\dot{+}(x, y) = (-x, \bar{y}\lambda)$ for any $(x, y) \in \Delta_{max}$ by Lemma 7(1). Further $\dot{+}$ is commutative modulo Δ_{min} by Lemma 7(3).
- (c) One checks easily that Δ is a normal subgroup of $(R^2, \dot{+})$. Further Δ_{max}/Δ is a right R -module with scalar multiplication $(a \dot{+} \Delta)x = (a \bullet x) \dot{+} \Delta$.
- (d) $\Delta^{-1} := \{(x, y) \in R^2 | (x, \bar{y}) \in \Delta\}$ is an odd form parameter for $(R, \bar{\cdot}, \lambda, \mu)$.
- (e) $\Lambda(\Delta) := \{x \in R | (0, x) \in \Delta\}$ is a form parameter for R (as defined in [9] (for the case that λ is central) or in [11] (general case)).

Next we define an odd form ideal of an odd form ring.

Definition 11. Let (R, Δ) be an odd form ring and I an involution invariant ideal of R . Set $J(\Delta) := \{y \in R | \exists z \in R : (y, z) \in \Delta\}$ and $\tilde{I} := \{x \in R | \overline{J(\Delta)}\mu x \subseteq I\}$. Further set

$$\Omega_{min}^I := \{(0, x - \bar{x}\lambda) | x \in I\} \dot{+} \Delta \bullet I$$

and

$$\Omega_{max}^I := \Delta \cap (\tilde{I} \times I).$$

An R -subquasimodule Ω of \mathfrak{H} lying between Ω_{min}^I and Ω_{max}^I is called a *relative odd form parameter for I* . Since Ω_{min}^I and Ω_{max}^I are R -subquasimodules of \mathfrak{H} , they are respectively the smallest and the largest relative odd form parameter for I . If Ω is a relative odd form parameter for I , then (I, Ω) is called an *odd form ideal of (R, Δ)* .

Remark 12. Let Ω be a relative odd form parameter for I .

- (a) Obviously \tilde{I} and $J(\Delta)$ are right ideals of R . Further $I \subseteq \tilde{I}$. Set $J(\Omega) := \{y \in R | \exists z \in R : (y, z) \in \Omega\}$. Then $J(\Omega)$ is a right ideal of R and

$$J(\Delta)I \subseteq J(\Omega) \subseteq J(\Delta) \cap \tilde{I}.$$

- (b) One checks easily that Ω is a normal subgroup of Δ . Further Ω_{max}^I/Ω is a right R -module with scalar multiplication $(a \dot{+} \Omega)x = (a \bullet x) \dot{+} \Omega$.
- (c) $\Omega^{-1} := \{(x, y) \in R^2 | (x, \bar{y}) \in \Omega\}$ is an odd form parameter for I with respect to the odd form ring (R, Δ^{-1}) .
- (d) $\Gamma(\Omega) := \{x \in R | (0, x) \in \Omega\}$ is a relative form parameter for I with respect to the form ring $(R, \Lambda(\Delta))$ (cf. [9] or [11]).

Definition 13. Let (R, Δ) be an odd form ring. Further let $Y \subseteq R$ and $Z \subseteq \Delta$ be subsets. Then we denote by $I(Y)$ the ideal of R generated by $Y \cup \bar{Y}$. It is called the *involution invariant ideal defined by Y* . Further set $\Omega(Y) := \Omega_{min}^{I(Y)}$. It is called the *relative odd form parameter defined by Y* and $(I(Y), \Omega(Y))$ is called the *odd form ideal defined by Y* . Set $Z_1 := \{x \in R | \exists y \in R : (x, y) \in Z\}$, $Z_2 := \{y \in R | \exists x \in R : (x, y) \in Z\}$ and $Z' := \overline{J(\Delta)}\mu Z_1 \cup Z_2$. We denote by $I(Z)$ the ideal of R generated by $Z' \cup \bar{Z'}$. It is called the *involution invariant ideal defined by Z* . Further set $\Omega(Z) := \Omega_{min}^{I(Z)} \dot{+} Z \bullet R$. $\Omega(Z)$ is called the *relative odd form parameter defined by Z* . One checks easily that $(I(Z), \Omega(Z))$ is an odd form ideal of (R, Δ) . It is called the *odd form ideal defined by Z* .

Remark 14. If $x \in R$ or $x \in \Delta$, we sometimes write $(I(x), \Omega(x))$ instead of $(I(\{x\}), \Omega(\{x\}))$.

3.2. Odd-dimensional unitary groups. Let (R, Δ) be an odd form ring and $n \in \mathbb{N}$. Fix the base $(e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1})$ of the right R -module $M := R^{2n+1}$ where e_i is the column whose i -th entry is 1 and whose other entries are 0 if $i > 0$, the column whose $(2n+2+i)$ -th entry is 1 and whose other entries are 0 if $i < 0$ and the column whose 0-th entry is 1 and whose other entries are 0 if $i = 0$. If

$u \in M$, then we call $(u_1, \dots, u_n, u_{-n}, \dots, u_{-1})^t \in R^{2n}$ *hyperbolic part of u* and denote it by u_{hb} . Define the maps

$$b : M \times M \rightarrow R$$

$$(u, v) \mapsto u^* \begin{pmatrix} 0 & 0 & p \\ 0 & \mu & 0 \\ p\lambda & 0 & 0 \end{pmatrix} v = \sum_{i=1}^n \bar{u}_i v_{-i} + \bar{u}_0 \mu v_0 + \sum_{i=-n}^{-1} \bar{u}_i \lambda v_{-i}$$

and

$$q : M \rightarrow R^2$$

$$u \mapsto (q_1(u), q_2(u)) := (u_0, u_{hb}^* \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} u_{hb}) = (u_0, \sum_{i=1}^n \bar{u}_i u_{-i})$$

where $u^* = \bar{u}^t$ and $p \in M_n(R)$ is the matrix with ones on the second diagonal and zeros elsewhere.

Lemma 15.

- (1) b is a λ -Hermitian form, i.e. b is biadditive, $b(ux, vy) = \bar{x}b(u, v)y \ \forall u, v \in M, x, y \in R$ and $b(u, v) = \overline{b(v, u)\lambda} \ \forall u, v \in M$.
- (2) $q(ux) = q(u) \bullet x \ \forall u \in M, x \in R$, $q(u + v) \equiv q(u) + q(v) + (0, b(u, v)) \pmod{\Delta_{min}} \ \forall u, v \in M$ and $tr(q(u)) = b(u, u) \ \forall u \in M$.

Proof. Straightforward computation. □

Definition 16. The group

$$U_{2n+1}(R, \Delta) := \{\sigma \in GL_{2n+1}(R) | b(\sigma u, \sigma v) = b(u, v) \ \forall u, v \in M \text{ and } q(\sigma u) \equiv q(u) \pmod{\Delta} \ \forall u \in M\}$$

is called *odd-dimensional unitary group*.

Remark 17.

(a) Clearly

$$\begin{aligned} b(\sigma u, \sigma u) &= b(u, u) \\ \Leftrightarrow tr(q(\sigma u)) &= tr(q(u)) \\ \Leftrightarrow tr(q(\sigma u) - q(u)) &= 0 \\ \Leftrightarrow q(\sigma u) - q(u) &\in \Delta_{max} \\ \Leftrightarrow q(\sigma u) &\equiv q(u) \pmod{\Delta_{max}} \end{aligned}$$

for any $\sigma \in M_{2n+1}(R)$ and $u \in M$. Hence

$$U_{2n+1}(R, \Delta_{max}) = \{\sigma \in GL_{2n+1}(R) | b(\sigma u, \sigma v) = b(u, v) \ \forall u, v \in M\}.$$

(b) Set $\Lambda := \Lambda(\Delta)$ and let $m, n \in \mathbb{N}$ such that $m \leq n$. There is an embedding of $U_{2m}(R, \Lambda)$ (as defined in [9] or [11]) into $U_{2n+1}(R, \Delta)$ namely

$$\phi_{2m}^{2n+1} : U_{2m}(R, \Lambda) \rightarrow U_{2n+1}(R, \Delta)$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B \\ 0 & e^{k \times k} & 0 \\ C & 0 & D \end{pmatrix}$$

where $A, B, C, D \in M_m(R)$ and $k = 1 + 2(n - m)$. There is also an embedding of $U_{2m+1}(R, \Delta)$ into $U_{2n+1}(R, \Delta)$ namely

$$\begin{aligned} \phi_{2m+1}^{2n+1} : U_{2m+1}(R, \Delta) &\rightarrow U_{2n+1}(R, \Delta) \\ \sigma &\mapsto \begin{pmatrix} e^{l \times l} & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & e^{l \times l} \end{pmatrix} \end{aligned}$$

where $l = n - m$.

- (c) The odd-dimensional unitary groups defined in this article are precisely Petrov's odd hyperbolic unitary groups with $V_0 = R$ (see [8]). Namely $U_{2n+1}(R, \Delta) = U_{2l}(R, \mathfrak{L})$ if $\bar{x} = -x^* \lambda$ (where $*$ denotes the anti-isomorphism on R used in this article), $V_0 = R$, $B_0(a, b) = \bar{a} \bar{1}^{-1} \mu b$, $\mathfrak{L} = \{(x, y) \in R^2 \mid (x, -y) \in \Delta\}$ and $l = n$.

Example 18.

- (1) Let (R, Λ) be a form ring. Choose μ arbitrary and set $\Delta := \{0\} \times \Lambda$. Then $U_{2n+1}(R, \Delta) \cong U_{2n}(R, \Lambda)$. In particular the groups $O_{2n}(R)$ and $Sp_{2n}(R)$ where R is a commutative ring and $GL_{2n}(R)$ where R is an arbitrary ring are examples of odd-dimensional unitary groups.
- (2) If $R = S \times S^{op}$ where S is a ring, $\bar{(x, y)} = (y, x)$, $\lambda = 1$, $\mu = 1$ and $\Delta = \Delta_{max}$, then $U_{2n+1}(R, \Delta) \cong GL_{2n+1}(S)$.
- (3) If R is commutative, $\bar{x} = x$, $\lambda = 1$, $\mu = 2$ and $\Delta = \{(x, -x^2) \mid x \in R\}$, then $U_{2n+1}(R, \Delta) = O_{2n+1}(R)$.
- (4) If R is commutative, $\bar{x} = x$, $\lambda = -1$, $\mu = 0$ and $\Delta = R \times R$, then $U_{2n+1}(R, \Delta)$ is the automorphism group of the bilinear form represented by

$$\begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ -p & 0 & 0 \end{pmatrix}.$$

It makes sense to denote this group by $Sp_{2n+1}(R)$ since it is the automorphism group of a skew-symmetric bilinear form of maximal possible rank (cf. [10]).

Definition 19. Define $\Theta_+ := \{1, \dots, n\}$, $\Theta_- := \{-n, \dots, -1\}$, $\Theta := \Theta_+ \cup \Theta_- \cup \{0\}$, $\Theta_{hb} := \Theta \setminus \{0\}$ and the map

$$\begin{aligned} \epsilon : \Theta_{hb} &\rightarrow \{\pm 1\} \\ i &\mapsto \begin{cases} 1, & \text{if } i \in \Theta_+, \\ -1, & \text{if } i \in \Theta_-. \end{cases} \end{aligned}$$

Lemma 20. Let $\sigma \in GL_{2n+1}(R)$. Then $\sigma \in U_{2n+1}(R, \Delta)$ if and only if the conditions (1) and (2) below hold.

(1)

$$\begin{aligned} \sigma'_{ij} &= \lambda^{-(\epsilon(i)+1)/2} \bar{\sigma}_{-j, -i} \lambda^{(\epsilon(j)+1)/2} \quad \forall i, j \in \Theta_{hb}, \\ \mu \sigma'_{0j} &= \bar{\sigma}_{-j, 0} \lambda^{(\epsilon(j)+1)/2} \quad \forall j \in \Theta_{hb}, \\ \sigma'_{i0} &= \lambda^{-(\epsilon(i)+1)/2} \bar{\sigma}_{0, -i} \mu \quad \forall i \in \Theta_{hb} \text{ and} \\ \mu \sigma'_{00} &= \bar{\sigma}_{00} \mu. \end{aligned}$$

(2)

$$q(\sigma_{*j}) \equiv (\delta_{0j}, 0) \pmod{\Delta} \quad \forall j \in \Theta.$$

Proof.

“ \Rightarrow ”:

Assume that $\sigma = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \in U_{2n+1}(R, \Delta)$ where $A, C, G, I \in M_n(R)$, $B, H \in R^n$, $D, F \in {}^nR$ and

$E \in R$. Then $b(\sigma u, \sigma v) = b(u, v) \forall u, v \in M$ and $q(\sigma u) \equiv q(v) \pmod{\Delta} \forall u \in M$. Let $\sigma^{-1} = \begin{pmatrix} A' & B' & C' \\ D' & E' & F' \\ G' & H' & I' \end{pmatrix}$

where A' has the same size as A , B' has the same size as B and so on. Clearly

$$\begin{aligned}
& b(\sigma u, \sigma v) = b(u, v) \quad \forall u, v \in M \\
& \Leftrightarrow \sigma^* \begin{pmatrix} 0 & 0 & p \\ 0 & \mu & 0 \\ p\lambda & 0 & 0 \end{pmatrix} \sigma = \begin{pmatrix} 0 & 0 & p \\ 0 & \mu & 0 \\ p\lambda & 0 & 0 \end{pmatrix} \\
& \Leftrightarrow \sigma^* \begin{pmatrix} 0 & 0 & p \\ 0 & \mu & 0 \\ p\lambda & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & p \\ 0 & \mu & 0 \\ p\lambda & 0 & 0 \end{pmatrix} \sigma^{-1} \\
& \Leftrightarrow \begin{pmatrix} A^* & D^* & G^* \\ B^* & E^* & H^* \\ C^* & F^* & I^* \end{pmatrix} \begin{pmatrix} 0 & 0 & p \\ 0 & \mu & 0 \\ p\lambda & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & p \\ 0 & \mu & 0 \\ p\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} A' & B' & C' \\ D' & E' & F' \\ G' & H' & I' \end{pmatrix} \\
& \Leftrightarrow \begin{pmatrix} G^*p\lambda & D^*\mu & A^*p \\ H^*p\lambda & E^*\mu & B^*p \\ I^*p\lambda & F^*\mu & C^*p \end{pmatrix} = \begin{pmatrix} pG' & pH' & pI' \\ \mu D' & \mu E' & \mu F' \\ p\lambda A' & p\lambda B' & p\lambda C' \end{pmatrix} \\
& \Leftrightarrow \begin{pmatrix} pG^*p\lambda & pD^*\mu & pA^*p \\ H^*p\lambda & E^*\mu & B^*p \\ \bar{\lambda}pI^*p\lambda & \bar{\lambda}pF^*\mu & \bar{\lambda}pC^*p \end{pmatrix} = \begin{pmatrix} G' & H' & I' \\ \mu D' & \mu E' & \mu F' \\ A' & B' & C' \end{pmatrix} \\
& \Leftrightarrow \begin{pmatrix} \bar{\lambda}pI^*p\lambda & \bar{\lambda}pF^*\mu & \bar{\lambda}pC^*p \\ H^*p\lambda & E^*\mu & B^*p \\ pG^*p\lambda & pD^*\mu & pA^*p \end{pmatrix} = \begin{pmatrix} A' & B' & C' \\ \mu D' & \mu E' & \mu F' \\ G' & H' & I' \end{pmatrix} \\
& \Leftrightarrow \text{condition (1)}.
\end{aligned}$$

Further $q(\sigma u) \equiv q(u) \pmod{\Delta} \forall u \in M$ implies condition (2), since $q(e_j) = (0, 0)$ for any $j \neq 0$ and $q(e_0) = (1, 0)$.

“ \Leftarrow ”:

Assume that (1) and (2) hold. As shown in “ \Rightarrow ”, (1) is equivalent to $b(\sigma u, \sigma v) = b(u, v) \forall u, v \in M$. It remains to show that $q(\sigma u) \equiv q(u) \pmod{\Delta} \forall u \in M$. But this follows from (1), (2) and Lemma 15. \square

3.3. The elementary subgroup.

Definition 21. If $i, j \in \Theta_{hb}$, $i \neq \pm j$ and $x \in R$, the element

$$T_{ij}(x) := e + xe^{ij} - \lambda^{(\epsilon(j)-1)/2} \bar{x} \lambda^{(1-\epsilon(i))/2} e^{-j, -i}$$

of $U_{2n+1}(R, \Delta)$ is called an *(elementary) short root matrix*. If $i \in \Theta_{hb}$ and $(x, y) \in \Delta^{-\epsilon(i)}$, the element

$$T_i(x, y) := e + xe^{0, -i} - \lambda^{-(1+\epsilon(i))/2} \bar{x} \mu e^{i0} + ye^{i, -i}$$

of $U_{2n+1}(R, \Delta)$ is called an *(elementary) extra short root matrix*. The extra short root matrices

$$T_i(0, y) = e + ye^{i, -i}$$

are called *(elementary) long root matrices*. The subgroup of $U_{2n+1}(R, \Delta)$ generated by the short and extra short root matrices is called *elementary subgroup* and is denoted by $EU_{2n+1}(R, \Delta)$. Let (I, Ω) denote an odd form ideal of (R, Δ) . A short root matrix $T_{ij}(x)$ is called *(I, Ω)-elementary* if $x \in I$. An extra short root matrix $T_i(x, y)$ is called *(I, Ω)-elementary* if $(x, y) \in \Omega^{-\epsilon(i)}$. The subgroup of $EU_{2n+1}(R, \Delta)$ generated

by the (I, Ω) -elementary short and extra short root matrices is called *preelementary subgroup of level (I, Ω)* and is denoted by $EU_{2n+1}(I, \Omega)$. Its normal closure in $EU_{2n+1}(R, \Delta)$ is called *elementary subgroup of level (I, Ω)* and is denoted by $EU_{2n+1}((R, \Delta), (I, \Omega))$.

Remark 22. Set $\Lambda := \Lambda(\Delta)$ and let $m \leq n$ be a natural number. One checks easily that $\phi_{2m}^{2n+1}(EU_{2m}(R, \Lambda)) \subseteq EU_{2n+1}(R, \Delta)$ and $\phi_{2m+1}^{2n+1}(EU_{2m+1}(R, \Delta)) \subseteq EU_{2n+1}(R, \Delta)$.

Lemma 23. *The relations*

$$T_{ij}(x) = T_{-j, -i}(-\lambda^{(\epsilon(j)-1)/2} \bar{x} \lambda^{(1-\epsilon(i))/2}) \quad (S1)$$

$$T_{ij}(x)T_{ij}(y) = T_{ij}(x+y) \quad (S2)$$

$$[T_{ij}(x), T_{kl}(y)] = e \text{ if } k \neq j, -i \text{ and } l \neq i, -j, \quad (S3)$$

$$[T_{ij}(x), T_{jk}(y)] = T_{ik}(xy) \text{ if } i \neq \pm k, \quad (S4)$$

$$[T_{ij}(x), T_{j, -i}(y)] = T_i(0, xy - \lambda^{(-1-\epsilon(i))/2} \bar{y} \bar{x} \lambda^{(1-\epsilon(i))/2}), \quad (S5)$$

$$T_i(x_1, y_1)T_i(x_2, y_2) = T_i((x_1, y_1) \dot{+}_{-\epsilon(i)} (x_2, y_2)), \quad (E1)$$

$$[T_i(x_1, y_1), T_j(x_2, y_2)] = T_{i, -j}(-\lambda^{-(1+\epsilon(i))/2} \bar{x}_1 \mu x_2) \text{ if } i \neq \pm j, \quad (E2)$$

$$[T_i(x_1, y_1), T_i(x_2, y_2)] = T_i(0, -\lambda^{-(1+\epsilon(i))/2} (\bar{x}_1 \mu x_2 - \bar{x}_2 \mu x_1)), \quad (E3)$$

$$[T_{ij}(x), T_k(y, z)] = e \text{ if } k \neq j, -i \text{ and} \quad (SE1)$$

$$[T_{ij}(x), T_j(y, z)] = T_{j, -i}(z \lambda^{(\epsilon(j)-1)/2} \bar{x} \lambda^{(1-\epsilon(i))/2}) T_i(y \lambda^{(\epsilon(j)-1)/2} \bar{x} \lambda^{(1-\epsilon(i))/2}, x z \lambda^{(\epsilon(j)-1)/2} \bar{x} \lambda^{(1-\epsilon(i))/2}) \quad (SE2)$$

hold true.

Proof. Straightforward computation. □

Definition 24. Let $i, j \in \Theta_{hb}$ such that $i \neq \pm j$. Define

$$\begin{aligned} P_{ij} &:= e - e^{ii} - e^{jj} - e^{-i, -i} - e^{-j, -j} + e^{ij} - e^{ji} + \lambda^{(\epsilon(i)-\epsilon(j))/2} e^{-i, -j} - \lambda^{(\epsilon(j)-\epsilon(i))/2} e^{-j, -i} \\ &= T_{ij}(1)T_{ji}(-1)T_{ij}(1) \in EU_{2n+1}(R, \Delta). \end{aligned}$$

It is easy to show that $(P_{ij})^{-1} = P_{ji}$.

Lemma 25. Let $x \in R$ and $i, j, k \in \Theta_{hb}$ such that $i \neq \pm j$ and $k \neq \pm i, \pm j$. Further let $(y, z) \in \Delta^{-\epsilon(i)}$. Then

- (1) $P_{ki}T_{ij}(x) = T_{kj}(x)$,
- (2) $P_{kj}T_{ij}(x) = T_{ik}(x)$ and
- (3) $P_{-k, -i}T_i(y, z) = T_k(y, \lambda^{(\epsilon(i)-\epsilon(k))/2} z)$.

Proof. Using the relations in Lemma 23 one gets

$$\begin{aligned} & P_{ki}T_{ij}(x) \\ &= T_{ki}(1)T_{ik}(-1)T_{ki}(1)T_{ij}(x) \\ &\stackrel{(S4)}{=} T_{ki}(1)T_{ik}(-1)(T_{kj}(x)T_{ij}(x)) \\ &\stackrel{(S3), (S4)}{=} T_{ki}(1)(T_{ij}(-x)T_{kj}(x)T_{ij}(x)) \\ &\stackrel{(S3), (S4)}{=} T_{kj}(-x)T_{ij}(-x)T_{kj}(x)T_{kj}(x)T_{ij}(x) \\ &\stackrel{(S2), (S3)}{=} T_{kj}(x). \end{aligned}$$

Further one gets

$$\begin{aligned}
& P_{kj} T_{ij}(x) \\
&= T_{kj}(1) T_{jk}(-1) T_{kj}(1) T_{ij}(x) \\
&\stackrel{(S3)}{=} T_{kj}(1) T_{jk}(-1) T_{ij}(x) \\
&\stackrel{(S4)}{=} T_{kj}(1) (T_{ik}(x) T_{ij}(x)) \\
&\stackrel{(S3),(S4)}{=} T_{ij}(-x) T_{ik}(x) T_{ij}(x) \\
&\stackrel{(S2),(S3)}{=} T_{ik}(x)
\end{aligned}$$

and

$$\begin{aligned}
& P_{-k,-i} T_i(y, z) \\
&= T_{-k,-i}(1) T_{-i,-k}(-1) T_{-k,-i}(1) T_i(y, z) \\
&\stackrel{(SE1)}{=} T_{-k,-i}(1) T_{-i,-k}(-1) T_i(y, z) \\
&\stackrel{(S1)}{=} T_{-k,-i}(1) T_{ki}(\lambda^{(\epsilon(i)-\epsilon(k))/2}) T_i(y, z) \\
&\stackrel{(SE2)}{=} T_{-k,-i}(1) (T_{i,-k}(z) T_k(y, \lambda^{(\epsilon(i)-\epsilon(k))/2} z) T_i(y, z)) \\
&\stackrel{(S1),(S5), (SE1),(SE2)}{=} T_i(0, -(z - \lambda^{(-1-\epsilon(i))/2} \bar{z} \lambda^{(1-\epsilon(i))/2})) T_{i,-k}(z) T_{k,-i}(-\lambda^{(\epsilon(i)-\epsilon(k))/2} z) \times \\
&\quad \times T_i(-y, z) T_k(y, \lambda^{(\epsilon(i)-\epsilon(k))/2} z) T_i(y, z) \\
&\stackrel{(S1),(E2)}{=} T_i(0, -(z - \lambda^{(-1-\epsilon(i))/2} \bar{z} \lambda^{(1-\epsilon(i))/2})) T_{k,-i}(-\lambda^{(\epsilon(-k)-1)/2} \bar{z} \lambda^{(1-\epsilon(i))/2}) \times \\
&\quad \times T_{k,-i}(-\lambda^{(\epsilon(i)-\epsilon(k))/2} z) T_i(-y, z) T_{k,-i}(-\lambda^{-(1+\epsilon(k))/2} \bar{y} \mu y) T_i(y, z) \times \\
&\quad \times T_k(y, \lambda^{(\epsilon(i)-\epsilon(k))/2} z) \\
&\stackrel{(S2),(E1), (SE1)}{=} T_k(y, \lambda^{(\epsilon(i)-\epsilon(k))/2} z).
\end{aligned}$$

□

3.4. Congruence subgroups. Until the end of this subsection (I, Ω) denotes an odd form ideal of (R, Δ) . If $\sigma \in M_{2n+1}(R)$, we call the matrix $(\sigma_{ij})_{i,j \in \Theta_{hb}} \in M_{2n}(R)$ *hyperbolic part of σ* and denote it by σ_{hb} . Further we denote the submodule $\{u \in M | u_0 \in J(\Delta)\}$ of M by $M(R, \Delta)$, the submodule $\{u \in M | u_i \in I \forall i \in \Theta_{hb}\}$ of M by $M(I)$ and the submodule $\{u \in M(I) | u_0 \in J(\Omega)\}$ of $M(I)$ by $M(I, \Omega)$. Note that if $\sigma \in U_{2n+1}(R, \Delta)$, $u \in M(R, \Delta)$, $v \in M(I)$ and $w \in M(I, \Omega)$ then $\sigma u \in M(R, \Delta)$ and $\sigma v \in M(I)$ but not necessarily $\sigma w \in M(I, \Omega)$. Define

$$I_0 := \{x \in R | xJ(\Delta) \subseteq I\}$$

and

$$\tilde{I}_0 := \{x \in R | \overline{J(\Delta)} \mu x \in I_0\}.$$

Then I_0 is a left ideal and \tilde{I}_0 an additive subgroup of R . If $J(\Delta) = R$ then $I_0 = I$ and $\tilde{I}_0 = \tilde{I}$. Let $u, v \in M$ and $\sigma, \tau \in U_{2n+1}((R, \Delta), (I, \Omega))$. We write $u \equiv v \pmod{I, \tilde{I}}$ (resp. $u \equiv v \pmod{I_0, \tilde{I}_0}$) if and only if $u_i \equiv v_i \pmod{I} \forall i \in \Theta_{hb}$ and $u_0 \equiv v_0 \pmod{\tilde{I}}$ (resp. $u_i \equiv v_i \pmod{I_0} \forall i \in \Theta_{hb}$ and $u_0 \equiv v_0 \pmod{\tilde{I}_0}$). We write $\sigma \equiv \tau \pmod{I, \tilde{I}, I_0, \tilde{I}_0}$ if and only if $\sigma_{*j} \equiv \tau_{*j} \pmod{I, \tilde{I}} \forall j \in \Theta_{hb}$ and $\sigma_{*0} \equiv \tau_{*0} \pmod{I_0, \tilde{I}_0}$. Further if $w, x \in M^t := {}^{2n}R$, we write $w \equiv x \pmod{I, I_0}$ (resp. $w \equiv x \pmod{\tilde{I}, \tilde{I}_0}$) if and only if $w_j \equiv x_j \pmod{I} \forall j \in \Theta_{hb}$ and $w_0 \equiv x_0 \pmod{I_0}$ (resp. $w_j \equiv x_j \pmod{\tilde{I}} \forall j \in \Theta_{hb}$ and $w_0 \equiv x_0 \pmod{\tilde{I}_0}$).

Definition 26. The subgroup

$$\{\sigma \in U_{2n+1}(R, \Delta) | \sigma_{hb} \equiv e_{hb}(\text{mod } I) \text{ and } q(\sigma u) \equiv q(u)(\text{mod } \Omega) \forall u \in M(R, \Delta)\}$$

of $U_{2n+1}(R, \Delta)$ is called *principal congruence subgroup of level (I, Ω)* and is denoted by $U_{2n+1}((R, \Delta), (I, \Omega))$.

Lemma 27. Let $u, v \in M$ such that $u \in M(I)$ or $v \in M(I)$. Then

$$q(u + v) \equiv q(u) \dot{+} q(v) \dot{+} (0, b(u, v))(\text{mod } \Omega_{\min}^I).$$

Proof. A straightforward computation shows that

$$q(u + v) \dot{-} (q(u) \dot{+} q(v) \dot{+} (0, b(u, v))) = (0, \sum_{i=1}^n \bar{v}_i u_{-i} - \overline{\sum_{i=1}^n \bar{v}_i u_{-i} \lambda}) \in \Omega_{\min}^I.$$

□

Lemma 28. Let $\sigma \in U_{2n+1}(R, \Delta)$. Then $\sigma \in U_{2n+1}((R, \Delta), (I, \Omega))$ if and only if the conditions (1) and (2) below hold.

(1) $\sigma_{hb} \equiv e_{hb}(\text{mod } I)$.

(2) $q(\sigma_{*j}) \in \Omega \forall j \in \Theta_{hb}$ and $(q(\sigma_{*0}) \dot{-} (1, 0)) \bullet x \in \Omega \forall x \in J(\Delta)$.

Proof.

“ \Rightarrow ”:

Assume that $\sigma \in U_{2n+1}((R, \Delta), (I, \Omega))$. Then (1) holds and $q(\sigma u) \dot{-} q(u) \in \Omega \forall u \in M$. Clearly $q(\sigma u) \dot{-} q(u) \in \Omega \forall u \in M$ implies (2), since $q(e_j) = (0, 0)$ for any $j \neq 0$ and $q(e_0) = (1, 0)$.

“ \Leftarrow ”:

Assume that (1) and (2) hold. We have to show that $q(\sigma u) \equiv q(u)(\text{mod } \Omega) \forall u \in M$. But this follows from (1), (2) and Lemma 27 (see the proof of Lemma 60). □

Remark 29. Let $\sigma \in U_{2n+1}(R, \Delta)$. Lemma 28 implies that $\sigma \in U_{2n+1}((R, \Delta), (I, \Omega_{\max}^I))$ if and only if the conditions (1) and (2) below hold.

(1) $\sigma_{hb} \equiv e_{hb}(\text{mod } I)$.

(2) $\sigma_{0*} \equiv e_0^t(\text{mod } \tilde{I}, \tilde{I}_0)$.

It follows from (2) that if $\sigma \in U_{2n+1}((R, \Delta), (I, \Omega_{\max}^I))$, then $\sigma_{*0} \equiv e_0(\text{mod } I_0, \tilde{I}_0)$ and therefore $\sigma \equiv e(\text{mod } I, \tilde{I}, I_0, \tilde{I}_0)$.

The subgroup $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$ defined below will be used to define the full congruence subgroup $CU_{2n+1}((R, \Delta), (I, \Omega))$.

Definition 30. The subgroup

$$\{\sigma \in U_{2n+1}(R, \Delta) | q(\sigma u) \equiv q(\sigma^{-1}u) \equiv q(u)(\text{mod } \Omega) \forall u \in M(I, \Omega)\}$$

of $U_{2n+1}(R, \Delta)$ is denoted by $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$.

Remark 31.

(a) Obviously $U_{2n+1}((R, \Delta), (I, \Omega)) \subseteq \tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$.

(b) $EU_{2n+1}(R, \Delta) \subseteq \tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$ (see [8, p.4760]).

(c) In many interesting situations, $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$ equals $U_{2n+1}(R, \Delta)$, e.g. in the situations (1)-(3) in Example 18. Further the equality holds if $\Omega = \Omega_{\max}^I$ or $J(\Omega) \subseteq I$ (true e.g. if $\Omega = \Omega_{\min}^I$). An example for a situation where $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega)) \neq U_{2n+1}(R, \Delta)$ can be found in the last section of this paper (see Example 66).

Lemma 32. $U_{2n+1}((R, \Delta), (I, \Omega))$ is a normal subgroup of $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$.

Proof. See [8, Proposition 5]. In Section 5 we will give another proof. \square

Remark 33. Lemma 28 implies that $EU_{2n+1}(I, \Omega) \subseteq U_{2n+1}((R, \Delta), (I, \Omega))$. By the previous lemma $U_{2n+1}((R, \Delta), (I, \Omega))$ is normalized by $EU_{2n+1}(R, \Delta)$ since $EU_{2n+1}(R, \Delta) \subseteq \tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$. It follows that $EU_{2n+1}((R, \Delta), (I, \Omega)) \subseteq U_{2n+1}((R, \Delta), (I, \Omega))$.

Definition 34. The subgroup

$$\{\sigma \in \tilde{U}_{2n+1}((R, \Delta), (I, \Omega)) \mid [\sigma, EU_{2n+1}(R, \Delta)] \subseteq U_{2n+1}((R, \Delta), (I, \Omega))\}$$

of $U_{2n+1}(R, \Delta)$ is called *full congruence subgroup of level (I, Ω)* and is denoted by $CU_{2n+1}((R, \Delta), (I, \Omega))$.

Remark 35.

(a) Let

$$\pi : \tilde{U}_{2n+1}((R, \Delta), (I, \Omega)) \rightarrow \tilde{U}_{2n+1}((R, \Delta), (I, \Omega)) / U_{2n+1}((R, \Delta), (I, \Omega))$$

be the canonical group homomorphism. Set $G := \pi(\tilde{U}_{2n+1}((R, \Delta), (I, \Omega)))$ and $E := \pi(EU_{2n+1}(R, \Delta))$. Then clearly

$$CU_{2n+1}((R, \Delta), (I, \Omega)) = \pi^{-1}(\text{Centralizer}_G E).$$

(b) Obviously $U_{2n+1}((R, \Delta), (I, \Omega)) \subseteq CU_{2n+1}((R, \Delta), (I, \Omega))$. If $EU_{2n+1}(R, \Delta)$ is normalized by $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$ (which for example is true if $n \geq 3$ and R is semilocal or quasifinite, see the next theorem), then $CU_{2n+1}((R, \Delta), (I, \Omega))$ is a normal subgroup of $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$.

Theorem 36 (V. A. Petrov). *If $n \geq 3$ and R is semilocal or quasifinite, then $EU_{2n+1}((R, \Delta), (I, \Omega))$ is a normal subgroup of $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$ and the standard commutator formulas*

$$\begin{aligned} & [CU_{2n+1}((R, \Delta), (I, \Omega)), EU_{2n+1}(R, \Delta)] \\ &= [EU_{2n+1}((R, \Delta), (I, \Omega)), EU_{2n+1}(R, \Delta)] \\ &= EU_{2n+1}((R, \Delta), (I, \Omega)) \end{aligned}$$

hold. In particular from the absolute case $(I, \Omega) = (R, \Delta)$, it follows that $EU_{2n+1}(R, \Delta)$ is perfect and normal in $U_{2n+1}(R, \Delta)$.

Proof. It follows from [8, Theorems 1 and 4] (semilocal case) resp. [8, corollary on page 4765] (quasifinite case), that $EU_{2n+1}((R, \Delta), (I, \Omega))$ is a normal subgroup of $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$ and

$$\begin{aligned} & [U_{2n+1}((R, \Delta), (I, \Omega)), EU_{2n+1}(R, \Delta)] \\ & \subseteq EU_{2n+1}((R, \Delta), (I, \Omega)). \end{aligned} \tag{36.1}$$

(note that in [8] the full congruence subgroup is defined a little differently). By [8, Proposition 4],

$$\begin{aligned} & [EU_{2n+1}((R, \Delta), (I, \Omega)), EU_{2n+1}(R, \Delta)] \\ &= EU_{2n+1}((R, \Delta), (I, \Omega)). \end{aligned} \tag{36.2}$$

Hence

$$\begin{aligned} & [CU_{2n+1}((R, \Delta), (I, \Omega)), EU_{2n+1}(R, \Delta)] \\ &= [EU_{2n+1}(R, \Delta), CU_{2n+1}((R, \Delta), (I, \Omega))] \\ &= [[EU_{2n+1}(R, \Delta), EU_{2n+1}(R, \Delta)], CU_{2n+1}((R, \Delta), (I, \Omega))] \\ & \subseteq EU_{2n+1}((R, \Delta), (I, \Omega)) \end{aligned} \tag{36.3}$$

by the definition of $CU_{2n+1}((R, \Delta), (I, \Omega))$, (36.1) and the three subgroups lemma. (36.2) and (36.3) imply the assertion of the theorem. \square

4. SANDWICH CLASSIFICATION OF E-NORMAL SUBGROUPS

In this section (R, Δ) denotes an odd form ring and $n \geq 3$ a natural number. A subgroup H of $U_{2n+1}(R, \Delta)$ normalized by $EU_{2n+1}(R, \Delta)$ is called *E-normal*. Let H be an E-normal subgroup of $U_{2n+1}(R, \Delta)$ and set

$$I := \{x \in R \mid T_{ij}(x) \in H \text{ for some } i, j \in \Theta_{hb}\}$$

and

$$\Omega := \{(x, y) \in \Delta \mid T_i(x, y) \in H \text{ for some } i \in \Theta_-\}.$$

Then (I, Ω) is an odd form ideal which is maximal with the property $EU_{2n+1}((R, \Delta), (I, \Omega)) \subseteq H$. It is called the *level of H* and H is called *E-normal subgroup of level (I, Ω)* .

We will show that if R semilocal or quasifinite, then a subgroup H of $U_{2n+1}(R, \Delta)$ is E-normal if and only if there is an odd form ideal (I, Ω) of (R, Δ) such that

$$EU_{2n+1}((R, \Delta), (I, \Omega)) \subseteq H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega)).$$

Further (I, Ω) is uniquely determined, namely it is the level of H .

The hardest part of the proof is showing that if H is E-normal, then $H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega))$ where (I, Ω) is the level of H . If R is semilocal we prove this by contradiction. Namely we show that if $H \not\subseteq CU_{2n+1}((R, \Delta), (I, \Omega))$, then one can produce an elementary matrix in H which is not (I, Ω) -elementary (cf. Lemma 4.3). If R is Noetherian and there is a subring C of $\text{Center}(R)$ with certain properties (i.a. $(C \setminus \mathfrak{m})^{-1}R$ is semilocal for any maximal ideal of C , cf. Section 4.2), then we use localisation to deduce the result from the semilocal case. If R is quasifinite, then we use the fact that EU_{2n+1} and U_{2n+1} commute with direct limits to deduce the result from the Noetherian case.

4.1. Semilocal case. In this subsection we assume that R is semilocal. Set $\Lambda := \Lambda(\Delta)$. Since R is semilocal, it satisfies the stable range condition SR_m and the Λ -stable range condition ΛS_m for any $m \in \mathbb{N}$ (follows from [7, Theorem 2.4] and [2, lemmas 3.2, 3.3, 3.4]). For the definition of SR_m and ΛS_m see [2]. We call a column vector $(u_1, \dots, u_m)^t \in R^m$ (*left*) *unimodular* iff there are $v_1, \dots, v_m \in R$ such that $\sum_{i=1}^m v_i u_i = 1$. Note that the map

$$\begin{aligned} * : R^m &\rightarrow {}^m R \\ u &\mapsto u^* := \bar{u}^t \end{aligned}$$

defines a bijection between left unimodular columns and right unimodular (in the sense of [2]) rows.

Lemma 37. *If $m \in \mathbb{N}$ and $(u_1, \dots, u_{m+1})^t$ is unimodular, then there is an $x \in R$ such that $(u_1 + xu_{m+1}, u_2, \dots, u_m)^t$ is unimodular.*

Proof. Let $v_1, \dots, v_{m+1} \in R$ such that $\sum_{i=1}^{m+1} v_i u_i = 1$. Clearly $(u_1, \sum_{i=2}^{m+1} v_i u_i)^t$ is unimodular. Hence $(\bar{u}_1, \overline{\sum_{i=2}^{m+1} v_i u_i})$ is right unimodular. Since R satisfies SR_m , there is a $\bar{y} \in R$ such that $\bar{u}_1 + (\overline{\sum_{i=2}^{m+1} v_i u_i}) \bar{y}$ is right invertible. It follows that $u_1 + y(\sum_{i=2}^{m+1} v_i u_i)$ is left invertible. Hence $(u_1 + yv_{m+1}u_{m+1}, u_2, \dots, u_m)^t$ is unimodular. \square

Definition 38. The subgroup of $EU_{2n+1}(R, \Delta)$ consisting of all $f \in EU_{2n+1}(R, \Delta)$ of the form

$$f = \begin{pmatrix} A & B & C \\ 0 & 1 & D \\ 0 & 0 & E \end{pmatrix}$$

where $A, B, C, D, E \in M_n(R)$ is denoted by $UEU_{2n+1}(R, \Delta)$. The subgroup of $UEU_{2n+1}(R, \Delta)$ consisting of all upper triangular matrices $f \in EU_{2n+1}(R, \Delta)$ with ones on the diagonal is denoted by $TEU_{2n+1}(R, \Delta)$.

Lemma 39. *Let $\sigma \in U_{2n+1}(R, \Delta)$. Then there is an $f \in TEU_{2n+1}(R, \Delta)$ such that $(f\sigma)_{11}$ is left invertible.*

Proof.

step 1

Let u be the first column of σ and v the first row of σ^{-1} . Since $vu = 1$,

$$(u_1, \dots, u_n, v_0 u_0, u_{-n}, \dots, u_{-1})^t$$

is unimodular. By Lemma 37, there is an $x \in R$ such that

$$(u_1 + xv_0 u_0, u_2, \dots, u_n, u_{-n}, \dots, u_{-1})^t$$

is unimodular. Since $\sigma \in U_{2n+1}(R, \Delta)$, $q(\sigma_{*, -1}) = (\sigma_{0, -1}, q_2(\sigma_{*, -1})) \in \Delta$. It follows that $(-\sigma_{0, -1}\bar{x}, \bar{x}q_2(\sigma_{*, -1})\bar{x}) \in \Delta$ and hence $a := (-\sigma_{0, -1}\bar{x}, \bar{x}q_2(\sigma_{*, -1})\bar{x}) \in \Delta^{-1}$. Set $f_1 := T_1(a) \in TEU_{2n+1}(R, \Delta)$ and $u^{(1)} := f_1 u$. One checks easily that

$$(u^{(1)})_{hb} = (u_1 + xv_0 u_0 + yu_{-1}, u_2, \dots, u_n, u_{-n}, \dots, u_{-1})^t$$

where $y = \bar{x}q_2(\sigma_{*, -1})\bar{x}$ (note that $v_0 = \sigma'_{10} = \bar{\lambda}\bar{\sigma}_{0, -1}\mu$ by Lemma 20). Therefore $(u^{(1)})_{hb}$ is unimodular.

step 2

Since $(u^{(1)})_{hb}$ is unimodular and R satisfies ΛS_{n-1} , there is a matrix

$$\rho = \begin{pmatrix} e & \gamma \\ 0 & e \end{pmatrix} \in EU_{2n}(R, \Lambda)$$

where $\gamma \in M_n(R)$ such that $(w_1, \dots, w_n)^t$ is unimodular where $w = \rho(u^{(1)})_{hb}$. Set

$$f_2 = \phi_{2n}^{2n+1}(\rho) = \begin{pmatrix} e & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & e \end{pmatrix} \in TEU_{2n+1}(R, \Delta)$$

(where ϕ_{2n}^{2n+1} is defined as in Remark 17(b)) and $u^{(2)} := f_2 u^{(1)}$. Then clearly $(u_1^{(2)}, \dots, u_n^{(2)})^t = (w_1, \dots, w_n)^t$ and hence $(u_1^{(2)}, \dots, u_n^{(2)})^t$ is unimodular.

step 3

Since $(u_1^{(2)}, \dots, u_n^{(2)})^t$ is unimodular and R satisfies SR_m for any $m \in \mathbb{N}$, there is an $f_3 \in TEU_{2n+1}(R, \Delta)$ such that if $f_3 u^{(2)} = u^{(3)}$, then $u_1^{(3)}$ is left invertible.

Hence if $f = f_3 f_2 f_1 \in TEU_{2n+1}(R, \Delta)$, then $(f\sigma)_{11}$ is left invertible. □

Lemma 40. *Let $\sigma \in U_{2n+1}(R, \Delta)$. Then there is a $f \in UEU_{2n+1}(R, \Delta)$ such that $((f\sigma)_{11}, (f\sigma)_{21}, \dots, (f\sigma)_{n1}) = (1, 0, \dots, 0)$ and $((f\sigma)_{12}, (f\sigma)_{22}, (f\sigma)_{32}, \dots, (f\sigma)_{n2}) = (0, 1, 0, \dots, 0)$.*

Proof.

step 1

Let u be the first column of σ . By steps 1 and 2 in the previous lemma, there is an $f_1 \in UEU_{2n+1}(R, \Delta)$ such that if $u^{(1)} = f_1 u$, then $(u_1^{(1)}, \dots, u_n^{(1)})^t$ is unimodular.

step 2

Since $(u_1^{(1)}, \dots, u_n^{(1)})^t$ is unimodular, there is an $f_2 \in UEU_{2n+1}(R, \Delta)$ such that if $f_2 u^{(1)} = u^{(2)}$, then $(u_1^{(2)}, \dots, u_n^{(2)})^t = (1, 0, \dots, 0)^t$ (see [4, Chapter V, (3.3)(1)]). Set $\tau := f_2 f_1 \sigma$. Then the first two columns of τ equal

$$\begin{pmatrix} 1 & F \\ 0 & G \\ C & H \\ D & I \\ E & J \end{pmatrix}$$

for some $C, E, F, H, J \in R$ and $D, G, I \in M_{(n-1) \times 1}(R)$.

step 3

Set $\epsilon_1 := \prod_{j=-n}^{-2} T_{j1}(-u_j^{(2)})$ and $u^{(3)} := \epsilon_1 u^{(2)}$. Then clearly $u^{(3)} = e_1 + e_0 x + e_{-1} y$ for some $x, y \in R$. Since $u^{(3)}$ is the first column of a unitary matrix, $(x, y) = q(u^{(3)}) \in \Delta$. Set $\epsilon_2 := T_{-1}(\cdot(x, y))$ and $u^{(4)} := \epsilon_2 u^{(3)}$. One checks easily that $u^{(4)} = e_1 = (1, 0, \dots, 0)^t$. Set $\epsilon := \epsilon_2 \epsilon_1$. Since the first column of $\epsilon \tau$ equals $u^{(4)} = e_1$, its last row equals e_{-1}^t (follows from Lemma 20). Hence the first two columns of $\epsilon \tau$ equal

$$\begin{pmatrix} 1 & F \\ 0 & G \\ 0 & H - CF \\ 0 & I - DF \\ 0 & 0 \end{pmatrix}.$$

It follows that $(G, z(H - CF), I - DF)^t$ is unimodular for some $z \in \overline{J(\Delta)}\mu$ (see step 1 in the previous lemma). Set $\xi := T_{12}(-F)$. Then the first two columns of $\tau \xi$ equal

$$\begin{pmatrix} 1 & 0 \\ 0 & G \\ C & H - CF \\ D & I - DF \\ E & J - EF \end{pmatrix}.$$

Since $(G, z(H - CF), I - DF)^t$ is unimodular, there is a $\zeta \in UEU_{2n-1}(R, \Delta)$ such that the first $n - 1$ coefficients of $\zeta(G, H - CF, I - DF)^t$ equal $(1, 0, \dots, 0)^t$ (see steps 1 and 2). Set $f_3 := \psi_{2n-1}^{2n+1}(\zeta) \in UEU_{2n+1}(R, \Delta)$. Then there are $C', H' \in R$ and $B', D', I' \in M_{(n-1) \times 1}(R)$ such that the first two columns of $f_3 \tau \xi$ equal

$$\begin{pmatrix} 1 & 0 \\ B' & G' \\ C' & H' \\ D' & I' \\ E & J - EF \end{pmatrix}$$

where $G' = (1, 0, \dots, 0)^t \in M_{(n-1) \times 1}$.

step 4

Set $f_4 := \prod_{j=2}^n T_{j1}(-B'_j) \in UEU_{2n+1}(R, \Delta)$. Then the first two columns of $f_4 f_3 \tau \xi$ equal

$$\begin{pmatrix} 1 & 0 \\ 0 & G' \\ C' & H' \\ D' & I' \\ E' & J' \end{pmatrix}$$

for some $E', J' \in R$. Hence the first two columns of $f_4 f_3 \tau$ equal

$$\begin{pmatrix} 1 & F \\ 0 & G' \\ C' & H'' \\ D' & I'' \\ E' & J'' \end{pmatrix}$$

for some $H'', J'' \in R$ and a $I'' \in M_{(n-1) \times 1}(R)$.

step 5

Set $f_5 := T_{12}(-F) \in UEU_{2n+1}(R, \Delta)$ and $f := f_5 \dots f_1 \in UEU_{2n+1}(R, \Delta)$. Then the first two columns of $f\sigma = f_5 f_4 f_3 \tau$ equal

$$\begin{pmatrix} 1 & 0 \\ 0 & G' \\ C' & H'' \\ D'' & I''' \\ E' & J'' \end{pmatrix}$$

for some $D'', I'' \in M_{(n-1) \times 1}(R)$. □

Lemma 41. *Let $\sigma \in U_{2n+1}(R, \Delta)$. Further let $i, j \in \Theta_{hb}$ such that $i \neq \pm j$, $x \in R$ and $(y, z) \in \Delta^{-\epsilon(i)}$. Set $\tau := [\sigma, T_{ij}(x)]$ and $\rho := [\sigma, T_i(y, z)]$. Further let $J(\sigma)$ denote the ideal generated by $\{\sigma_{kl}, \sigma'_{kl} | k, l \in \Theta_{hb}, k \neq l\} \cup \{\bar{a}\mu\sigma_{0l}, \bar{a}\mu\sigma'_{0l} | a \in J(\Delta), l \in \Theta_{hb}\}$ and $J'(\sigma)$ the left ideal generated by $\{\sigma_{k0}, \sigma'_{k0} | k \in \Theta_{hb}\}$. Then*

$$q(\tau_{*k}) = (\delta_{0k}, 0) \dot{+} q(\sigma_{*i}) \bullet x\sigma'_{jk} \dot{+} q(\sigma_{*, -j}) \bullet \tilde{x}\sigma'_{-i, k} \dot{+} (0, y_k - \bar{y}_k\lambda),$$

if $k \neq j, -i$,

$$q(\tau_{*j}) = q(\sigma_{*i}) \bullet x\sigma'_{jj} \dot{+} q(\sigma_{*, -j}) \bullet \tilde{x}\sigma'_{-i, j} \dot{+} q(\tau_{*i}) \bullet (-x) \dot{+} (0, y_j - \bar{y}_j\lambda)$$

and

$$q(\tau_{*, -i}) = q(\sigma_{*i}) \bullet x\sigma'_{j, -i} \dot{+} q(\sigma_{*, -j}) \bullet \tilde{x}\sigma'_{-i, -i} \dot{+} q(\tau_{*, -j}) \bullet (-\tilde{x}) \dot{+} (0, y_{-i} - \bar{y}_{-i}\lambda)$$

where $\tilde{x} = -\lambda^{(\epsilon(j)-1)/2} \bar{x} \lambda^{(1-\epsilon(i))/2}$, $y_k \in J(\sigma) \forall k \in \Theta_{hb}$ and $y_0 \in J'(\sigma)$.

Further

$$\begin{aligned} q(\rho_{*k}) = & (q(\sigma_{*0}) \dot{-} (1, 0)) \bullet y\sigma'_{-i, k} \dot{+} q(\sigma_{*i}) \bullet \hat{y}\sigma'_{0k} \dot{+} q(\sigma_{*i}) \bullet z\sigma'_{-i, k} \\ & \dot{+} a_k \bullet \sigma'_{-i, k} \dot{+} (0, z_k - \bar{z}_k\lambda), \end{aligned}$$

if $k \neq 0, -i$,

$$\begin{aligned} q(\rho_{*0}) = & (1, 0) \dot{+} (q(\sigma_{*0}) \dot{-} (1, 0)) \bullet y\sigma'_{-i, 0} \dot{+} q(\sigma_{*i}) \bullet \hat{y}\sigma'_{00} \dot{+} q(\sigma_{*i}) \bullet z\sigma'_{-i, 0} \\ & \dot{+} q(\rho_{*i}) \bullet (-\hat{y}) \dot{+} a_0 \bullet \sigma'_{-i, 0} \dot{+} (0, z_0 - \bar{z}_0\lambda) \end{aligned}$$

and

$$\begin{aligned} q(\rho_{*, -i}) = & (q(\sigma_{*0}) \dot{-} (1, 0)) \bullet y\sigma'_{-i, -i} \dot{+} q(\sigma_{*i}) \bullet \hat{y}\sigma'_{0, -i} \dot{+} q(\sigma_{*i}) \bullet z\sigma'_{-i, -i} \\ & \dot{+} (q(\sigma_{*0}) \dot{-} (1, 0)) \bullet (-y\sigma'_{-i, 0}y) \dot{+} q(\sigma_{*i}) \bullet (-\hat{y}\sigma'_{00}y) \dot{+} q(\sigma_{*i}) \bullet (-z\sigma'_{-i, 0}y) \\ & \dot{+} q(\rho_{*i}) \bullet \hat{z} \dot{+} a_{-k} \bullet (\sigma'_{-i, -i} - 1) \dot{+} b \dot{+} c \bullet (-\sigma'_{-i, 0}y) \dot{+} d \dot{+} (0, z_{-i} - \bar{z}_{-i}\lambda) \end{aligned}$$

where $\hat{y} = -\lambda^{-(1+\epsilon(i))/2} \bar{y} \mu$, $\hat{z} = \lambda^{-(1+\epsilon(i))/2} \bar{z} \lambda^{(1-\epsilon(i))/2}$,

$$a_k = \begin{cases} (y, \lambda^{(\epsilon(i)+1)/2} z), & \text{if } k \in \Theta \setminus \Theta_-, \\ (y, \bar{z} \lambda^{(1-\epsilon(i))/2}), & \text{if } k \in \Theta_-, \end{cases}$$

$$b = \begin{cases} (0, \bar{z}(\sigma'_{-1, -1} - 1) - \overline{\bar{z}(\sigma'_{-1, -1} - 1)\lambda}), & \text{if } i \in \Theta_+, \\ (0, z(\sigma'_{-1, -1} - 1) - \overline{z(\sigma'_{-1, -1} - 1)\lambda}), & \text{if } i \in \Theta_-, \end{cases}$$

$$c = \begin{cases} (y, \lambda z), & \text{if } i \in \Theta_+, \\ (y, z), & \text{if } i \in \Theta_-, \end{cases}$$

$$d = \begin{cases} (0, \bar{\sigma}'_{-i, -i} \bar{y} \bar{\sigma}_{00} \mu y - \overline{\bar{\sigma}'_{-i, -i} \bar{y} \bar{\sigma}_{00} \mu y \lambda}), & \text{if } i \in \Theta_+, \\ 0, & \text{if } i \in \Theta_-, \end{cases}$$

$z_k \in J(\sigma) \forall k \in \Theta_{hb}$ and $z_0 \in J(\sigma) + J'(\sigma)$.

Proof. Straightforward computation. □

Lemma 42. *Let I be an involution invariant ideal of R and $h \in U_{2n+1}(R, \Delta) \setminus CU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Then either*

(1) *there are an $f \in EU_{2n+1}(R, \Delta)$ and an $x \in R$ such that*

$$[{}^f h, T_{1,-2}(x)] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$$

or

(2) *there are an $f \in EU_{2n+1}(R, \Delta)$, an $x \in R$, a $k \in \Theta_{hb}$ and a $(y, z) \in \Delta^{-\epsilon(k)}$ such that*

$$[{}^f [h, T_k(y, z)], T_{1,-2}(x)] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I)).$$

Proof.

case 1 Assume that $[h, T_{ij}(x)] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ for some $x \in R$ and $i, j \in \Theta_{hb}$ such that $i \neq \pm j$. By Lemma 25 there is an $f \in EU_{2n+1}(R, \Delta)$ such that ${}^f T_{ij}(x) = T_{1,-2}(x)$. Hence $[{}^f h, T_{1,-2}(x)] = [{}^f [h, T_{ij}(x)], T_{1,-2}(x)] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Thus (1) holds.

case 2 Assume that $[h, T_{ij}(x)] \in U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ for any $x \in R$ and $i, j \in \Theta_{hb}$ such that $i \neq \pm j$. One checks easily that in this case $h_{ij} \in I$ for any $i, j \in \Theta_{hb}$ such that $i \neq j$ and $h_{0j} \in \tilde{I}$ for any $j \in \Theta_{hb}$. Since $h \notin CU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$, there are a $k \in \Theta_{hb}$ and a $(y, z) \in \Delta^{-\epsilon(k)}$ such that $[h, T_k(y, z)] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. One checks easily that $[h, T_k(y, z)]_{0,-k} \notin \tilde{I}$ or $[h, T_k(y, z)]_{k,-k} \notin I$. It follows that $[[h, T_k(y, z)], T_{1k}(1)] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Hence $[h, T_k(y, z)] \notin CU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Applying case 1 to $[h, T_k(y, z)]$ we get (2). \square

Lemma 43. *Let (I, Ω) be an odd form ideal and H an E -normal subgroup of level (I, Ω) . Then $H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega))$.*

Proof. Suppose $H \not\subseteq CU_{2n+1}((R, \Delta), (I, \Omega))$. We will show that it follows that H contains an elementary matrix which is not (I, Ω) -elementary. This of course contradicts the assumption, that H is of level (I, Ω) . Choose an $h \in H \setminus CU_{2n+1}((R, \Delta), (I, \Omega))$. The proof is divided into three parts, I, II and III. In Part I we assume that $h \notin CU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ and $n > 3$, in Part II we assume that $h \notin CU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ and $n = 3$ and in Part III we assume that $h \in CU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$.

Part I Assume that $h \notin CU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ and $n > 3$.

By Lemma 42, either (1) or (2) in Lemma 42 holds.

case 1 Assume that (1) in Lemma 42 holds.

Then there are an $f_0 \in EU_{2n+1}(R, \Delta)$ and an $x \in R$ such that $[{}^{f_0} h, T_{1,-2}(x)] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Set $g_0 := T_{1,-2}(x)$. By Lemma 40 there is an $f_1 \in UEU_{2n+1}(R, \Delta)$ such that the first n coefficients of $f_1({}^{f_0} h)_{*1}$ equal $(1, 0, \dots, 0)^t$ and the first n coefficients of $f_1({}^{f_0} h)_{*2}$ equal $(0, 1, 0, \dots, 0)^t$. Set $\sigma := f_1[{}^{f_0} h, g_0]$. Clearly

$$\begin{aligned} \sigma &= f_1[{}^{f_0} h, g_0] \\ &= f_1(g_0^{-1} + ({}^{f_0} h)_{*1} x (({}^{f_0} h)^{-1})_{-2,*} g_0^{-1} - ({}^{f_0} h)_{*2} \bar{\lambda} \bar{x} (({}^{f_0} h)^{-1})_{-1,*} g_0^{-1}) \\ &= f_1 g_0^{-1} + f_1(({}^{f_0} h)_{*1} x (({}^{f_0} h)^{-1})_{-2,*} g_0^{-1}) - f_1(({}^{f_0} h)_{*2} \bar{\lambda} \bar{x} (({}^{f_0} h)^{-1})_{-1,*} g_0^{-1}). \end{aligned}$$

Hence the i -th row of σ equals the i -th row of $f_1 g_0^{-1}$ for any $i \in \{3, \dots, n\}$. Since $f_1 \in UEU_{2n+1}(R, \Delta)$, $f_1 g_0^{-1}$ has the form

$$f_1 g_0^{-1} = \left(\begin{array}{c|c|c} e^{n \times n} & 0 & * \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & e^{n \times n} \end{array} \right).$$

Therefore there are $A, B, C, D \in M_n(R)$, $t, u \in M_{n \times 1}(R)$, $v, w \in M_{1 \times n}(R)$, $z \in R$, $A_1, B_2 \in M_{n-2}(R)$, $A_2, B_1 \in M_{(n-2) \times 2}(R)$, $B_3 \in M_2(R)$, $B_4 \in M_{2 \times (n-2)}(R)$ and $t_1 \in M_{(n-2) \times 1}(R)$ such that

$$\sigma = \left(\begin{array}{c|c|c} A & t & B \\ \hline v & z & w \\ \hline C & u & D \end{array} \right) = \left(\begin{array}{c|c|c} A_1 & A_2 & t_1 \\ \hline 0 & e^{2 \times 2} & 0 \\ \hline v & z & w \\ \hline C & u & D \end{array} \right).$$

Since $[^{f_0}h, g_0] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ and $U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ is a normal subgroup of $U_{2n+1}(R, \Delta)$, $\sigma = ^{f_1}[^{f_0}h, g_0] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Hence, by Remark 29, $\sigma_{hb} \not\equiv e_{hb}(\text{mod } I)$ or $\sigma_{0*} \not\equiv e_0^t(\text{mod } \tilde{I}, \tilde{I}_0)$.

case 1.1 Assume that $A \not\equiv e(\text{mod } I) \vee v \not\equiv 0(\text{mod } \tilde{I}) \vee C \not\equiv 0(\text{mod } I) \vee D \not\equiv e(\text{mod } I)$.

We will show that it follows that $A \not\equiv e(\text{mod } I) \vee v \not\equiv 0(\text{mod } \tilde{I}) \vee C \not\equiv 0(\text{mod } I)$. Assume that $A \equiv e(\text{mod } I) \wedge v \equiv 0(\text{mod } \tilde{I}) \wedge C \equiv 0(\text{mod } I)$. Then clearly

$$\delta_{ij} = (\sigma^{-1}\sigma)_{ij} = \sum_{k=1}^{-1} \sigma'_{ik} \sigma_{kj} \equiv \sigma'_{ij}(\text{mod } I) \quad \forall i, j \in \Theta_+$$

where δ_{ij} is the Kronecker delta (note that $\sigma'_{i0} \sigma_{0j} \in I$ since $\sigma'_{i0} \in \overline{J(\Delta)\mu}$ by Lemma 20 and $\sigma_{0j} \in \tilde{I}$). It follows from Lemma 20, that $\sigma_{ij} \equiv \delta_{ij}(\text{mod } I) \quad \forall i, j \in \Theta_-$, i.e. $D \equiv e(\text{mod } I)$. Since this is a contradiction, $A \not\equiv e(\text{mod } I) \vee v \not\equiv 0(\text{mod } J) \vee C \not\equiv 0(\text{mod } I)$. Hence there is a $j \in \Theta_+$ such that $\sigma_{*j} \not\equiv e_j(\text{mod } I, \tilde{I})$.

case 1.1.1 Assume there is a $j \in \{1, \dots, n-2\}$ such that $\sigma_{*j} \not\equiv e_j(\text{mod } I, \tilde{I})$.

Set $g_1 := T_{j, -(n-1)}(1)$. Clearly the $(n-1)$ -th row of

$$\begin{aligned} & [\sigma, g_1] \\ &= (e + \sigma_{*j} \sigma'_{-(n-1),*} - \sigma_{*,n-1} \bar{\lambda} \sigma'_{-j,*}) g_1^{-1} \\ &= (e + \begin{matrix} 1 \\ n \\ 0 \\ -n \\ -1 \end{matrix} \begin{pmatrix} \sigma_{1j} \\ \vdots \\ \sigma_{n-2,j} \\ 0 \\ 0 \\ \sigma_{0j} \\ \sigma_{-n,j} \\ \vdots \\ \sigma_{-1,j} \end{pmatrix} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\sigma}_{-1,n-1} \lambda \dots \bar{\sigma}_{-n,n-1} \lambda & \bar{\sigma}_{0,n-1} \mu & 0 & 1 & \bar{\sigma}_{n-2,n-1} \dots \bar{\sigma}_{1,n-1} \end{pmatrix}) \\ & - \begin{matrix} 1 \\ n \\ 0 \\ -n \\ -1 \end{matrix} \begin{pmatrix} \sigma_{1,n-1} \\ \vdots \\ \sigma_{n-2,n-1} \\ 1 \\ 0 \\ \sigma_{0,n-1} \\ \sigma_{-n,n-1} \\ \vdots \\ \sigma_{-1,n-1} \end{pmatrix} \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\sigma}_{-1,j} \lambda \dots \bar{\sigma}_{-n,j} \lambda & \bar{\sigma}_{0j} \mu & 0 & 0 & \bar{\sigma}_{n-2,j} \dots \bar{\sigma}_{1j} \end{pmatrix}) g_1^{-1} \end{aligned}$$

is not congruent to e_{n-1}^t modulo I, I_0 since $\sigma_{*j} \not\equiv e_j \pmod{I, \tilde{I}}$. Hence $[\sigma, g_1] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Clearly the n -th row of $[\sigma, g_1]$ equals e_n^t and its $(-n)$ -th column equals e_{-n} . Set $f_2 := P_{1n}$ and $\tau := f_2[\sigma, g_1]$. Then the first row of τ equals e_1^t and its last column equals e_{-1} . Hence there are $E \in M_{2n-1}(R)$, $\alpha \in M_{(2n-1) \times 1}(R)$, $\beta \in M_{1 \times (2n-1)}(R)$, $\gamma \in R$, $A', B', C', D' \in M_{n-1}(R)$, $t', u', \alpha_1, \alpha_3 \in M_{(n-1) \times 1}(R)$, $v', w', \beta_1, \beta_3 \in M_{1 \times (n-1)}(R)$ and $\alpha_2, \beta_2, z' \in R$ such that

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & E & 0 \\ \gamma & \beta & 1 \end{pmatrix} = \left(\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ \alpha_1 & A' & t' & B' & 0 \\ \alpha_2 & v' & z' & w' & 0 \\ \alpha_3 & C' & u' & D' & 0 \\ \gamma & \beta_1 & \beta_2 & \beta_3 & 1 \end{array} \right).$$

Since $U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ is normal, $\tau \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$.

case 1.1.1.1 Assume that $E \notin U_{2n-1}((R, \Delta), (I, \Omega_{max}^I))$.

case 1.1.1.1.1 Assume that $A' \not\equiv e \pmod{I} \vee B' \not\equiv 0 \pmod{I} \vee C' \not\equiv 0 \pmod{I} \vee D' \not\equiv e \pmod{I}$.

Then there are $i, j \in \{2, \dots, -2\} \setminus \{0\}$ such that $\tau_{ij} \not\equiv \delta_{ij} \pmod{I}$. Set $g_2 := T_{-1,i}(1)$. Then $\omega := [\tau^{-1}, g_2]$ has the form

$$\omega = \begin{pmatrix} 1 & 0 & 0 \\ * & e^{(2n-1) \times (2n-1)} & 0 \\ * & w & 1 \end{pmatrix}$$

where $w = (w_2, \dots, w_{-2}) = (\tau_{i2}, \dots, \tau_{i,-2}) - e_i^t$. Clearly $\tau_{ij} \not\equiv \delta_{ij} \pmod{I}$ implies that $w_j \notin I$. Choose a $k \neq 0, \pm 1, \pm j$ and set $g_3 := T_{jk}(1)$, $g_4 := T_{k,-j}(1)$ and $g_5 := T_{-1,-j}(w_j)$. Note that g_5 is not (I, Ω) -elementary since $w_j \notin I$. One checks easily that

$$\begin{aligned} & [[[f_2([f_1[f_0 h, g_0], g_1]^{-1}), g_2], g_3], g_4] \\ &= [[[f_2([f_1[f_0 h, g_0], g_1]^{-1}), g_2], g_3], g_4] \\ &= [[\omega, g_3], g_4] \\ &= g_5. \end{aligned}$$

Hence $g_5 \in H$, since H is E-normal.

case 1.1.1.1.2 Assume that $A' \equiv e \pmod{I} \wedge B' \equiv 0 \pmod{I} \wedge C' \equiv 0 \pmod{I} \wedge D' \equiv e \pmod{I}$ and $v' \not\equiv 0 \pmod{\tilde{I}} \vee w' \not\equiv 0 \pmod{\tilde{I}}$.

Then there is a $j \in \{2, \dots, -2\} \setminus \{0\}$ such that $\tau_{0j} \notin \tilde{I}$. By the definition of \tilde{I} there is an $a \in J(\Delta)$ such that $\bar{a}\mu\tau_{0j} \notin I$. Choose a $b \in R$ such that $(a, b) \in \Delta$ and set $g_2 := T_{-1}(\cdot(a, b))$. Then $\omega := [\tau^{-1}, g_2]$ has the form

$$\omega = \begin{pmatrix} 1 & 0 & 0 \\ * & e^{(2n-1) \times (2n-1)} & 0 \\ * & w & 1 \end{pmatrix}$$

where $w = (w_2, \dots, w_{-2}) = \bar{a}\mu((\tau_{02}, \dots, \tau_{0,-2}) - e_0^t)$. Clearly $w_j = \bar{a}\mu\tau_{0j} \notin I$. One can proceed as in case 1.1.1.1.1

case 1.1.1.1.3 Assume that $A' \equiv e \pmod{I} \wedge B' \equiv 0 \pmod{I} \wedge C' \equiv 0 \pmod{I} \wedge D' \equiv e \pmod{I} \wedge v' \equiv 0 \pmod{\tilde{I}} \wedge w' \equiv 0 \pmod{\tilde{I}}$.

Since $E \notin U_{2n-1}((R, \Delta), (I, \Omega_{max}^I))$, it follows that $z' = \tau_{00} \not\equiv 1 \pmod{\tilde{I}_0}$. By the definition of \tilde{I}_0 there is an $a \in J(\Delta)$ such that $\bar{a}\mu(\tau_{00} - 1) \notin I_0$. Choose a $b \in R$ such that $(a, b) \in \Delta$ and set $g_2 := T_{-1}(\cdot(a, b))$.

Then $\omega := [\tau^{-1}, g_2]$ has the form

$$\omega = \begin{pmatrix} 1 & 0 & 0 \\ * & e^{(2n-1) \times (2n-1)} & 0 \\ * & w & 1 \end{pmatrix}$$

where $w = (w_2, \dots, w_{-2}) = \bar{a}\mu((\tau_{02}, \dots, \tau_{0,-2}) - e_0^t)$. Clearly $w_k \in I \ \forall k \in \{2, \dots, -2\} \setminus \{0\}$ and $w_0 \notin I_0$. By the definition of I_0 there is a $c \in J(\Delta)$ such that $w_0 c \notin I$. Choose a $d \in R$ such that $(c, d) \in \Delta$ and set $g_3 := T_{-2}(c, d)$. Then $[\omega, g_3]$ has the form

$$[\omega, g_3] = \begin{pmatrix} 1 & 0 & 0 \\ * & e^{(2n-1) \times (2n-1)} & 0 \\ * & w' & 1 \end{pmatrix}$$

where $w' \in M_{1 \times (2n-1)}(R)$ and $w'_2 = w_0 c - w_{-2} \bar{d} \lambda$. Clearly $w'_2 \notin I$. One can proceed as in case 1.1.1.1.1.

case 1.1.1.2 Assume that $E \in U_{2n-1}((R, \Delta), (I, \Omega_{max}^I))$ and $\alpha_1 \equiv 0 \pmod{I}$.

Set $\xi_1 := \prod_{i=2}^n T_{i1}(-\sigma_{i1}) \in EU_{2n+1}(I, \Omega) \subseteq H$ and $\kappa := \xi_1 \tau$. Then

$$\kappa = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & E & 0 \\ \gamma' & \beta' & 1 \end{pmatrix} = \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A' & t' & B' & 0 & \\ \hline \alpha_2 & v' & z' & w' & 0 & \\ \alpha_3 & C' & u' & D' & 0 & \\ \hline \gamma' & \beta'_1 & \beta'_2 & \beta'_3 & 1 & \end{array} \right)$$

for some $\beta' \in M_{1 \times (2n-1)}$, $\gamma', \beta'_2 \in R$ and $\beta'_1, \beta'_3 \in M_{1 \times (n-1)}(R)$. Clearly $\beta'_3 \equiv 0 \pmod{I}$ since $0 = b(e_1, e_j) = b(\kappa_{*1}, \kappa_{*j}) \equiv \kappa_{-1,j} \pmod{I} \ \forall j \in \{-n, \dots, -2\}$. Further $\kappa \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ since $\tau \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ and $\xi_1 \in EU_{2n+1}(I, \Omega) \subseteq H$.

case 1.1.1.2.1 Assume that there is a $j \in \{3, \dots, n\}$ such that $\kappa_{-1,j} \notin I$.

Set $f_{21} := T_{12}(-1)$ and $\omega := f_{21} \kappa$. Then ω has the form

$$\omega = \left(\begin{array}{cc|cc|cc} 1 & \epsilon & * & * & * & \\ 0 & A' & t'' & B'' & * & \\ \hline \eta_2 & v'' & z'' & w'' & * & \\ \hline \eta_3 & C'' & u'' & D'' & * & \\ \hline \gamma'' & \beta''_1 & \beta''_2 & \beta''_3 & * & \end{array} \right)$$

where $\epsilon \in M_{1 \times (n-1)}(R)$, B'' has the same size as B' , C'' has the same size as C' and so on. Furthermore $\epsilon \equiv 0 \pmod{I}$ and $\omega_{-2,j} \equiv \kappa_{-1,j} \pmod{I}$. Set $\xi_2 := \prod_{k=2}^n T_{1k}(-\omega_{1k}) \in EU_{2n+1}(I, \Omega) \subseteq H$ and $\rho := \omega \xi_2$. Then ρ has the form

$$\rho = \left(\begin{array}{cc|cc|cc} 1 & 0 & * & * & * & \\ 0 & A' & t''' & B''' & * & \\ \hline \eta'_2 & v''' & z''' & w''' & * & \\ \hline \eta'_3 & C''' & u''' & D''' & * & \\ \hline \gamma''' & \beta'''_1 & \beta'''_2 & \beta'''_3 & * & \end{array} \right)$$

where B''' has the same size as B'' , C''' has the same size as C'' and so on. Further $\rho_{-2,j} \equiv \omega_{-2,j} \equiv \kappa_{-1,j}(\text{mod } I)$. Since $\kappa_{-1,j} \notin I$, it follows that $\rho_{-2,j} \notin I$. Set $g_2 := T_{2,-j}(1)$. Then

$$\begin{aligned}
& [\rho, g_2] \\
& = (e + \rho_{*2}\rho'_{-j,*} - \rho_{*j}\bar{\lambda}\rho'_{-2,*})g_2^{-1} \\
& = (e + \begin{matrix} n \\ 0 \\ -n \\ -1 \end{matrix} \begin{pmatrix} 0 \\ \rho_{22} \\ \vdots \\ \rho_{n2} \\ \rho_{02} \\ \rho_{-n,2} \\ \vdots \\ \rho_{-1,2} \end{pmatrix} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\rho}_{-1,j}\lambda & \dots & \bar{\rho}_{-n,j}\lambda & \bar{\rho}_{0j}\mu & \bar{\rho}_{nj} & \dots & \bar{\rho}_{2j} & 0 \end{pmatrix} \\
& \quad - \begin{matrix} n \\ 0 \\ -n \\ -1 \end{matrix} \begin{pmatrix} 0 \\ \rho_{2j} \\ \vdots \\ \rho_{nj} \\ \rho_{0j} \\ \rho_{-n,j} \\ \vdots \\ \rho_{-1,j} \end{pmatrix} \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\rho}_{-1,2}\lambda & \dots & \bar{\rho}_{-n,2}\lambda & \bar{\rho}_{02}\mu & \bar{\rho}_{n2} & \dots & \bar{\rho}_{22} & 0 \end{pmatrix})g_2^{-1}.
\end{aligned}$$

Clearly $[\rho, g_2]_{1*} = e_1^t$, $[\rho, g_2]_{*, -1} = e_{-1}$ and $[\rho, g_2]_{22} = 1 + \rho_{22}\bar{\rho}_{-2,j}\lambda - \rho_{2j}\bar{\lambda}\bar{\rho}_{-2,2}\lambda$. Since $E \in U_{2n-1}((R, \Delta), (I, \Omega_{max}^I))$, $A' \equiv e(\text{mod } I)$. Hence $\rho_{22} \equiv 1(\text{mod } I)$ and $\rho_{2j} \in I$. It follows that $[\rho, g_2]_{22} \not\equiv 1(\text{mod } I)$ since $\rho_{-2,j} \notin I$. One can proceed as in case 1.1.1.1.

case 1.1.1.2.2 Assume that $\kappa_{-1,2} \notin I$.

Set $f_{21} := T_{13}(-1)$ and $\omega := f_{21}\kappa$. Then ω has the form

$$\omega = \left(\begin{array}{cc|cc} 1 & \epsilon & * & * & * \\ 0 & A' & t'' & B'' & * \\ \hline \eta_2 & v'' & z'' & w'' & * \\ \hline \eta_3 & C'' & u'' & D'' & * \\ \gamma'' & \beta_1'' & \beta_2'' & \beta_3'' & * \end{array} \right)$$

where $\epsilon \in M_{1 \times (n-1)}(R)$, B'' has the same size as B' , C'' has the same size as C' and so on. Furthermore $\epsilon \equiv 0(\text{mod } I)$ and $\omega_{-3,2} \equiv \kappa_{-1,2}(\text{mod } I)$. Set $\xi_2 := \prod_{k=2}^n T_{1k}(-\omega_{1k}) \in EU_{2n+1}(I, \Omega) \subseteq H$ and $\rho := \omega\xi_2$. Then ρ has the form

$$\rho = \left(\begin{array}{cc|cc} 1 & 0 & * & * & * \\ 0 & A' & t''' & B''' & * \\ \hline \eta_2' & v''' & z''' & w''' & * \\ \hline \eta_3' & C''' & u''' & D''' & * \\ \gamma''' & \beta_1''' & \beta_2''' & \beta_3''' & * \end{array} \right)$$

where B''' has the same size as B'' , C''' has the same size as C'' and so on. Further $\rho_{-3,2} \equiv \omega_{-3,2} \equiv \kappa_{-1,2}(\text{mod } I)$. Since $\kappa_{-1,2} \notin I$, it follows that $\rho_{-3,2} \notin I$. Set $g_2 := T_{3,-2}(1)$. Then

$$\begin{aligned}
& [\rho, g_2] \\
&= (e + \rho_{*3}\rho'_{-2,*} - \rho_{*2}\bar{\lambda}\rho'_{-3,*})g_2^{-1} \\
&= (e + \begin{matrix} n \\ 0 \\ -n \\ -1 \end{matrix} \begin{pmatrix} 0 \\ \rho_{23} \\ \vdots \\ \rho_{n3} \\ \rho_{03} \\ \rho_{-n,3} \\ \vdots \\ \rho_{-1,3} \end{pmatrix} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\rho}_{-1,2}\lambda & \dots & \bar{\rho}_{-n,2}\lambda & \bar{\rho}_{02}\mu & \bar{\rho}_{n2} & \dots & \bar{\rho}_{22} & 0 \end{pmatrix} \\
&\quad - \begin{matrix} n \\ 0 \\ -n \\ -1 \end{matrix} \begin{pmatrix} 0 \\ \rho_{22} \\ \vdots \\ \rho_{n2} \\ \rho_{02} \\ \rho_{-n,2} \\ \vdots \\ \rho_{-1,2} \end{pmatrix} \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\rho}_{-1,3}\lambda & \dots & \bar{\rho}_{-n,3}\lambda & \bar{\rho}_{03}\mu & \bar{\rho}_{n3} & \dots & \bar{\rho}_{23} & 0 \end{pmatrix})g_2^{-1}.
\end{aligned}$$

Clearly $[\rho, g_2]_{1*} = e_1^t$, $[\rho, g_2]_{*, -1} = e_{-1}$ and $[\rho, g_2]_{33} = 1 + \rho_{33}\bar{\rho}_{-3,2}\lambda - \rho_{32}\bar{\lambda}\bar{\rho}_{-3,3}\lambda$. Since $E \in U_{2n-1}((R, \Delta), (I, \Omega_{max}^I))$, $A' \equiv e(\text{mod } I)$. Hence $\rho_{33} \equiv 1(\text{mod } I)$ and $\rho_{32} \in I$. It follows that $[\rho, g_2]_{33} \not\equiv 1(\text{mod } I)$ since $\rho_{-3,2} \notin I$. One can proceed as in case 1.1.1.1.

case 1.1.1.2.3 Assume that $\kappa_{-1,j} \in I \forall j \in \{2, \dots, n\}$ and $\kappa_{-1,1} \notin I$.

Then $\beta'_1 \equiv 0(\text{mod } I)$ and $\gamma' \notin I$. By Lemma 20, $\kappa'_{j1} \in I \forall j \in \{-n, \dots, -2\}$ since $\kappa_{-1,j} \in I \forall j \in \{2, \dots, n\}$. It follows that $\kappa_{j1} \in I \forall j \in \{-n, \dots, -2\}$ since $\kappa_{j*}\kappa'_{*1} = 0 \forall j \in \{-n, \dots, -2\}$. Hence $\alpha_3 \equiv 0(\text{mod } I)$. Set $f_{21} := T_{12}(-1)$ and $\omega := f_{21}\kappa$. Then ω has the form

$$\omega = \left(\begin{array}{cc|cc|cc} 1 & \epsilon & * & * & * & * \\ 0 & A' & t'' & B'' & * & * \\ \hline \eta_2 & v'' & z'' & w'' & * & * \\ \hline \eta_3 & C'' & u'' & D'' & * & * \\ \hline \gamma'' & \beta''_1 & \beta''_2 & \beta''_3 & * & * \end{array} \right)$$

where $\epsilon \in M_{1 \times (n-1)}(R)$, B'' has the same size as B' , C'' has the same size as C' and so on. Furthermore $\epsilon \equiv 0(\text{mod } I)$ and $\omega_{-2,2} = \kappa_{-2,2} + \kappa_{-1,2} + \kappa_{-2,1} + \kappa_{-1,1} \equiv \kappa_{-1,1}(\text{mod } I)$. Set $\xi_2 := \prod_{k=2}^n T_{1k}(-\omega_{1k}) \in$

$EU_{2n+1}(I, \Omega) \subseteq H$ and $\rho := \omega \xi_2$. Then ρ has the form

$$\rho = \left(\begin{array}{cc|cc|c} 1 & 0 & * & * & * \\ 0 & A' & t''' & B''' & * \\ \hline \eta'_2 & v''' & z''' & w''' & * \\ \hline \eta'_3 & C''' & u''' & D''' & * \\ \gamma''' & \beta'_1 & \beta'_2 & \beta'_3 & * \end{array} \right)$$

where B''' has the same size as B'' , C''' has the same size as C'' and so on. Further $\rho_{-2,2} \equiv \omega_{-2,2} \equiv \kappa_{-1,1}(\text{mod } I)$. Since $\kappa_{-1,1} \notin I$, it follows that $\rho_{-2,2} \notin I$. Set $g_2 := T_{2,-3}(1)$. Then

$$\begin{aligned} & [\rho, g_2] \\ &= (e + \rho_{*2} \rho'_{-3,*} - \rho_{*3} \bar{\lambda} \rho'_{-2,*}) g_2^{-1} \\ &= (e + \begin{matrix} 1 \\ n \\ 0 \\ -n \\ -1 \end{matrix} \begin{pmatrix} 0 \\ \rho_{22} \\ \vdots \\ \rho_{n2} \\ \rho_{02} \\ \rho_{-n,2} \\ \vdots \\ \rho_{-1,2} \end{pmatrix} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\rho}_{-1,3}\lambda & \dots & \bar{\rho}_{-n,3}\lambda & \bar{\rho}_{03}\mu & \bar{\rho}_{n3} & \dots & \bar{\rho}_{23} & 0 \end{pmatrix} \\ & \quad - \begin{matrix} n \\ 0 \\ -n \\ -1 \end{matrix} \begin{pmatrix} 0 \\ \rho_{23} \\ \vdots \\ \rho_{n3} \\ \rho_{03} \\ \rho_{-n,3} \\ \vdots \\ \rho_{-1,3} \end{pmatrix} \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\rho}_{-1,2}\lambda & \dots & \bar{\rho}_{-n,2}\lambda & \bar{\rho}_{02}\mu & \bar{\rho}_{n2} & \dots & \bar{\rho}_{22} & 0 \end{pmatrix}) g_2^{-1}. \end{aligned}$$

Clearly $[\rho, g_2]_{1*} = e_1^t$, $[\rho, g_2]_{*, -1} = e_{-1}$ and $[\rho, g_2]_{32} = 1 + \rho_{32} \bar{\rho}_{-2,3} \lambda - \rho_{33} \bar{\lambda} \bar{\rho}_{-2,2} \lambda$. Since $E \notin U_{2n-1}((R, \Delta), (I, \Omega_{max}^I))$, $A' \equiv e(\text{mod } I)$. Hence $\rho_{33} \equiv 1(\text{mod } I)$ and $\rho_{32} \in I$. It follows that $[\rho, g_2]_{32} \not\equiv 1(\text{mod } I)$ since $\rho_{-2,2} \notin I$. One can proceed as in case 1.1.1.1.

case 1.1.1.2.4 Assume that $\kappa_{-1,j} \in I \forall j \in \{1, \dots, n\}$.

Then $\beta'_1 \equiv 0(\text{mod } I)$ and $\gamma' \in I$. It follows that $\alpha_3 \equiv 0(\text{mod } I)$ (see case 1.1.1.2.3). Hence $\alpha_2 \notin \tilde{I}$ since $\kappa \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. It follows that $\beta'_2 \notin I_0$ (consider $b(\kappa_{*1}, \kappa_{*0})$). By the definition of I_0 there is an $a \in J(\Delta)$ such that $\beta'_2 a \notin I$. Choose a $b \in R$ such that $(a, b) \in \Delta$ and set $f_{21} := T_{-3}(a, b)$ and $\omega := f_{21} \kappa$. Then ω has the form

$$\omega = \begin{pmatrix} 1 & 0 & 0 \\ \eta & E'' & 0 \\ \gamma'' & \beta'' & 1 \end{pmatrix} = \left(\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & A'' & t'' & B'' & 0 \\ \hline \eta_2 & v'' & z'' & w'' & 0 \\ \hline \eta_3 & C'' & u'' & D'' & 0 \\ \gamma'' & \beta''_1 & \beta''_2 & \beta''_3 & 1 \end{array} \right)$$

where E'' has the same size as E , η has the same size as α and so on. Further $\omega_{-1,3} \equiv -\beta'_2 a \pmod{I}$ and therefore $\omega_{-1,3} \notin I$. Hence one can proceed as in case 1.1.1.1 or 1.1.1.2.1.

case 1.1.1.3 Assume that $E \in U_{2n-1}((R, \Delta), (I, \Omega_{max}^I))$ and $\alpha_1 \not\equiv 0 \pmod{I}$.

Then $\beta_3 \not\equiv 0 \pmod{I}$ (consider $b(\tau_{*1}, \tau_{*j})$ for $j \in \{-n, \dots, -2\}$). Hence there is an $j \in \{-n, \dots, -2\}$ such that $\tau_{-1,j} \notin I$. Choose an $i \in \{2, \dots, n\} \setminus \{-j\}$ and set $g_2 := T_{ji}(1)$ and $\omega := [\tau, g_2]$. Then

$$\begin{aligned} & \omega \\ &= [\tau, g_2] \\ &= (e + \tau_{*j} \tau'_{i*} - \tau_{*, -i} \lambda \tau'_{-j, *}) g_2^{-1} \\ &= (e + \begin{pmatrix} 0 \\ \tau_{2j} \\ \vdots \\ \tau_{nj} \\ \tau_{0j} \\ \tau_{-n,j} \\ \vdots \\ \tau_{-1,j} \end{pmatrix} \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\tau}_{-1, -i} \lambda & \dots & \bar{\tau}_{-n, -i} \lambda & \bar{\tau}_{0, -i} \mu & \bar{\tau}_{n, -i} & \dots & \bar{\tau}_{2, -i} & 0 \end{pmatrix} \\ & \quad - \begin{pmatrix} 0 \\ \tau_{2, -i} \\ \vdots \\ \tau_{n, -i} \\ \tau_{0, -i} \\ \tau_{-n, -i} \\ \vdots \\ \tau_{-1, -i} \end{pmatrix} \lambda \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\tau}_{-1, j} \lambda & \dots & \bar{\tau}_{-n, j} \lambda & \bar{\tau}_{0j} \mu & \bar{\tau}_{nj} & \dots & \bar{\tau}_{2j} & 0 \end{pmatrix}) g_2^{-1}. \end{aligned}$$

Clearly $\omega_{1*} = e_1^t$ and $\omega_{*, -1} = e_{-1}$. Hence ω has the form

$$\omega = \begin{pmatrix} 1 & 0 & 0 \\ \eta & E'' & 0 \\ \gamma'' & \beta'' & 1 \end{pmatrix} = \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ \eta_1 & A'' & t'' & B'' & 0 & \\ \hline \eta_2 & v'' & z'' & w'' & 0 & \\ \eta_3 & C'' & u'' & D'' & 0 & \\ \hline \gamma'' & \beta_1'' & \beta_2'' & \beta_3'' & 1 & \end{array} \right)$$

where E'' has the same size as E , η has the same size as α and so on. Clearly

$$\omega_{-1,i} = \tau_{-1,j} \bar{\lambda} \bar{\tau}_{-i, -i} \lambda - \tau_{-1, -i} \bar{\tau}_{-i, j} \lambda - (\tau_{-1,j} \bar{\lambda} \bar{\tau}_{-j, -i} - \tau_{-1, -i} \bar{\tau}_{-j, j}).$$

Since $\tau_{-i,j}, \tau_{-j, -i}, \tau_{-j,j} \in I$, $\tau_{-i, -i} \equiv 1 \pmod{I}$ and $\tau_{-1,j} \notin I$, it follows that $\omega_{-1,i} \notin I$ and hence $\omega \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Further $\eta_1 \equiv 0 \pmod{I}$. Thus one can proceed as in case 1.1.1.1 or 1.1.1.2.

case 1.1.2 Assume $\sigma_{*j} \equiv e_j \pmod{I, \tilde{I}} \forall j \in \{1, \dots, n-2\}$, $\sigma_{*, n-1} \not\equiv e_{n-1} \pmod{I, \tilde{I}}$ and $\sigma_{1, n-1} \in I$.

Set $g_1 := T_{1,-(n-1)}(1)$ and consider the first row of

$$\begin{aligned}
& [\sigma, g_1] \\
& = (e + \sigma_{*1} \sigma'_{-(n-1),*} - \sigma_{*,n-1} \bar{\lambda} \sigma'_{-1,*}) g_1^{-1} \\
& = (e + \begin{matrix} 1 & & & & & \\ & n & & & & \\ & 0 & & & & \\ & -n & & & & \\ & -1 & & & & \end{matrix} \begin{pmatrix} \sigma_{11} \\ \vdots \\ \sigma_{n-2,1} \\ 0 \\ 0 \\ \sigma_{01} \\ \sigma_{-n,1} \\ \vdots \\ \sigma_{-1,1} \end{pmatrix} \begin{pmatrix} 1 & n & 0 & -n & & -1 \\ \bar{\sigma}_{-1,n-1} \lambda \dots \bar{\sigma}_{-n,n-1} \lambda & \bar{\sigma}_{0,n-1} \mu & 0 & 1 & \bar{\sigma}_{n-2,n-1} \dots \bar{\sigma}_{1,n-1} \end{pmatrix}) \\
& - \begin{matrix} 1 & & & & & \\ & n & & & & \\ & 0 & & & & \\ & -n & & & & \\ & -1 & & & & \end{matrix} \begin{pmatrix} \sigma_{1,n-1} \\ \vdots \\ \sigma_{n-2,n-1} \\ 1 \\ 0 \\ \sigma_{0,n-1} \\ \sigma_{-n,n-1} \\ \vdots \\ \sigma_{-1,n-1} \end{pmatrix} \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & & -1 \\ \bar{\sigma}_{-1,1} \lambda \dots \bar{\sigma}_{-n,1} \lambda & \bar{\sigma}_{01} \mu & 0 & 0 & \bar{\sigma}_{n-2,1} \dots \bar{\sigma}_{11} \end{pmatrix}) g_1^{-1}
\end{aligned}$$

which equals

$$\begin{aligned}
& \begin{pmatrix} 1 & n & 0 & -n & & -1 \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \\
& + \sigma_{11} \begin{pmatrix} 1 & n & 0 & -n & & -1 \\ \bar{\sigma}_{-1,n-1} \lambda \dots \bar{\sigma}_{-n,n-1} \lambda & \bar{\sigma}_{0,n-1} \mu & 0 & 1 & \bar{\sigma}_{n-2,n-1} \dots \bar{\sigma}_{1,n-1} \end{pmatrix} \\
& - \sigma_{1,n-1} \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & & -1 \\ \bar{\sigma}_{-1,1} \lambda \dots \bar{\sigma}_{-n,1} \lambda & \bar{\sigma}_{01} \mu & 0 & 0 & \bar{\sigma}_{n-2,1} \dots \bar{\sigma}_{11} \end{pmatrix} \\
& + \begin{pmatrix} 1 & n & 0 & -n & & -1 \\ 0 & \dots & 0 & 0 & 0 & x_1 & 0 & \dots & 0 & x_2 \end{pmatrix}
\end{aligned}$$

for some $x_1, x_2 \in R$. It is clearly not congruent to e_1^t modulo I, I_0 since $\sigma_{11} \equiv 1 \pmod{I}$, $\sigma_{*,n-1} \not\equiv e_{n-1} \pmod{I, \tilde{I}}$ and $\sigma_{1,n-1} \in I$. Hence $[\sigma, g_1] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Clearly $[\sigma, g_1]_{n*} = e_n^t$ and $[\sigma, g_1]_{*, -n} = e_{-n}$. Set $f_2 := P_{1n}$ and $\tau := f_2[\sigma, g_1]$. Then $\tau_{1*} = e_1^t$ and $\tau_{*, -1} = e_{-1}$. Since $U_{2n+1}((R, \Delta),$

(I, Ω_{max}^I) is normal, $\tau \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. One can proceed now as in case 1.1.1 (τ has the same properties as the matrix τ in case 1.1.1).

case 1.1.3 Assume $\sigma_{*j} \equiv e_j \pmod{I, \tilde{I}} \forall j \in \{1, \dots, n-2\}$ and $\sigma_{1,n-1} \notin I$. Set $g_1 := T_{2, -(n-1)}(1)$ and consider the second row of

$$\begin{aligned}
& [\sigma, g_1] \\
& = (e + \sigma_{*2} \sigma'_{-(n-1),*} - \sigma_{*,n-1} \bar{\lambda} \sigma'_{-2,*}) g_1^{-1} \\
& = (e + \begin{matrix} 1 \\ n \\ 0 \\ -n \\ -1 \end{matrix} \begin{pmatrix} \sigma_{12} \\ \vdots \\ \sigma_{n-2,2} \\ 0 \\ 0 \\ \sigma_{02} \\ \sigma_{-n,2} \\ \vdots \\ \sigma_{-1,2} \end{pmatrix} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\sigma}_{-1,n-1} \lambda \dots \bar{\sigma}_{-n,n-1} \lambda & \bar{\sigma}_{0,n-1} \mu & 0 & 1 & \bar{\sigma}_{n-2,n-1} \dots \bar{\sigma}_{1,n-1} \end{pmatrix}) \\
& - \begin{matrix} 1 \\ n \\ 0 \\ -n \\ -1 \end{matrix} \begin{pmatrix} \sigma_{1,n-1} \\ \vdots \\ \sigma_{n-2,n-1} \\ 1 \\ 0 \\ \sigma_{0,n-1} \\ \sigma_{-n,n-1} \\ \vdots \\ \sigma_{-1,n-1} \end{pmatrix} \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\sigma}_{-1,2} \lambda \dots \bar{\sigma}_{-n,2} \lambda & \bar{\sigma}_{02} \mu & 0 & 0 & \bar{\sigma}_{n-2,2} \dots \bar{\sigma}_{12} \end{pmatrix}) g_1^{-1}
\end{aligned}$$

which equals

$$\begin{aligned}
& \begin{pmatrix} 1 & & n & 0 & -n & & -1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \\
& + \sigma_{22} \begin{pmatrix} 1 & & n & 0 & -n & & -1 \\ \bar{\sigma}_{-1,n-1} \lambda \dots \bar{\sigma}_{-n,n-1} \lambda & \bar{\sigma}_{0,n-1} \mu & 0 & 1 & \bar{\sigma}_{n-2,n-1} \dots \bar{\sigma}_{1,n-1} \end{pmatrix} \\
& - \sigma_{2,n-1} \bar{\lambda} \begin{pmatrix} 1 & & n & 0 & -n & & -1 \\ \bar{\sigma}_{-1,2} \lambda \dots \bar{\sigma}_{-n,2} \lambda & \bar{\sigma}_{02} \mu & 0 & 0 & \bar{\sigma}_{n-2,2} \dots \bar{\sigma}_{12} \end{pmatrix} \\
& + \begin{pmatrix} 1 & & n & 0 & -n & & -1 \\ 0 & \dots & 0 & 0 & 0 & x_1 & 0 & \dots & 0 & x_2 & 0 \end{pmatrix}
\end{aligned}$$

for some $x_1, x_2 \in R$. Its last entry clearly does not lie in I . Hence $[\sigma, g_1] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Clearly $[\sigma, g_1]_{n*} = e_n^t$ and $[\sigma, g_1]_{*, -n} = e_{-n}$. Set $f_2 := P_{1n}$ and $\tau := f_2[\sigma, g_1]$. Then $\tau_{1*} = e_1^t$ and $\tau_{*, -1} = e_{-1}$. Since $U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ is normal, $\tau \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. One can proceed now as in case 1.1.1.

case 1.1.4 Assume $\sigma_{*j} \equiv e_j \pmod{I, \tilde{I}} \forall j \in \{1, \dots, n-1\}$, $\sigma_{*n} \not\equiv e_n \pmod{I, \tilde{I}}$ and $\sigma_{1n} \in I$. Set $g_1 := T_{1, -n}(1)$ and consider the first row of

$$\begin{aligned}
& [\sigma, g_1] \\
& = (e + \sigma_{*1}\sigma'_{-n,*} - \sigma_{*n}\bar{\lambda}\sigma'_{-1,*})g_1^{-1} \\
& = (e + \begin{matrix} 1 & & & & \\ & n & & & \\ & 0 & & & \\ & -n & & & \\ & & & & -1 \end{matrix} \begin{pmatrix} \sigma_{11} \\ \vdots \\ \sigma_{n-2,1} \\ 0 \\ 0 \\ \sigma_{01} \\ \sigma_{-n,1} \\ \vdots \\ \sigma_{-1,1} \end{pmatrix} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\sigma}_{-1,n}\lambda \dots \bar{\sigma}_{-n,n}\lambda & \bar{\sigma}_{0n}\mu & 1 & 0 & \bar{\sigma}_{n-2,n} \dots \bar{\sigma}_{1n} \end{pmatrix}) \\
& - \begin{matrix} 1 & & & & \\ & n & & & \\ & 0 & & & \\ & -n & & & \\ & & & & -1 \end{matrix} \begin{pmatrix} \sigma_{1n} \\ \vdots \\ \sigma_{n-2,n} \\ 0 \\ 1 \\ \sigma_{0,n} \\ \sigma_{-n,n} \\ \vdots \\ \sigma_{-1,n} \end{pmatrix} \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\sigma}_{-1,1}\lambda \dots \bar{\sigma}_{-n,1}\lambda & \bar{\sigma}_{01}\mu & 0 & 0 & \bar{\sigma}_{n-2,1} \dots \bar{\sigma}_{11} \end{pmatrix}) g_1^{-1}
\end{aligned}$$

which equals

$$\begin{aligned}
& \begin{pmatrix} 1 & n & 0 & -n & -1 \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \\
& + \sigma_{11} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\sigma}_{-1,n}\lambda \dots \bar{\sigma}_{-n,n}\lambda & \bar{\sigma}_{0n}\mu & 1 & 0 & \bar{\sigma}_{n-2,n} \dots \bar{\sigma}_{1n} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& -\sigma_{1n}\bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\sigma}_{-1,1}\lambda \dots \bar{\sigma}_{-n,1}\lambda & \bar{\sigma}_{01}\mu & 0 & 0 & \bar{\sigma}_{n-2,1} \dots \bar{\sigma}_{11} \end{pmatrix} \\
& + \begin{pmatrix} 1 & n & 0 & -n & -1 \\ 0 & \dots & 0 & 0 & x_1 & 0 & 0 & \dots & 0 & x_2 \end{pmatrix}
\end{aligned}$$

for some $x_1, x_2 \in R$. It is clearly not congruent to e_1^t modulo I, I_0 since $\sigma_{11} \equiv 1 \pmod{I}$, $\sigma_{*n} \not\equiv e_n \pmod{I, \tilde{I}}$ and $\sigma_{1n} \in I$. Hence $[\sigma, g_1] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Clearly $[\sigma, g_1]_{n-1,*} = e_{n-1}^t$ and $[\sigma, g_1]_{*,-(n-1)} = e_{-(n-1)}$. Set $f_2 := P_{1,n-1}$ and $\tau := f_2[\sigma, g_1]$. Then $\tau_{1*} = e_1^t$ and $\tau_{*, -1} = e_{-1}$. Since $U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ is normal, $\tau \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. One can proceed now as in case 1.1.1.

case 1.1.5 Assume $\sigma_{*j} \equiv e_j \pmod{I, \tilde{I}} \forall j \in \{1, \dots, n-1\}$, $\sigma_{1n} \notin I$. Set $g_1 := T_{2,-n}(1)$ and consider the second row of

$$\begin{aligned}
& [\sigma, g_1] \\
& = (e + \sigma_{*2}\sigma'_{-n,*} - \sigma_{*n}\bar{\lambda}\sigma'_{-2,*})g_1^{-1} \\
& = (e + \begin{pmatrix} 1 & \sigma_{12} \\ & \vdots \\ & \sigma_{n-2,2} \\ & 0 \\ n & 0 \\ 0 & \sigma_{02} \\ -n & \sigma_{-n,2} \\ & \vdots \\ -1 & \sigma_{-1,2} \end{pmatrix} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\sigma}_{-1,n}\lambda \dots \bar{\sigma}_{-n,n}\lambda & \bar{\sigma}_{0n}\mu & 1 & 0 & \bar{\sigma}_{n-2,n} \dots \bar{\sigma}_{1n} \end{pmatrix}) \\
& - \begin{pmatrix} 1 & \sigma_{1n} \\ & \vdots \\ & \sigma_{n-2,n} \\ & 0 \\ n & 1 \\ 0 & \sigma_{0n} \\ -n & \sigma_{-n,n} \\ & \vdots \\ -1 & \sigma_{-1,n} \end{pmatrix} \bar{\lambda} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ \bar{\sigma}_{-1,2}\lambda \dots \bar{\sigma}_{-n,2}\lambda & \bar{\sigma}_{02}\mu & 0 & 0 & \bar{\sigma}_{n-2,2} \dots \bar{\sigma}_{12} \end{pmatrix}) g_1^{-1}
\end{aligned}$$

which equals

$$\begin{aligned}
& \begin{pmatrix} 1 & & n & 0 & -n & & & -1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \\
& + \sigma_{22} \begin{pmatrix} 1 & & n & 0 & -n & & & -1 \\ \bar{\sigma}_{-1,n}\lambda \dots \bar{\sigma}_{-n,n}\lambda & \bar{\sigma}_{0n}\mu & 1 & 0 & \bar{\sigma}_{n-2,n} \dots \bar{\sigma}_{1n} \end{pmatrix} \\
& - \sigma_{2n}\bar{\lambda} \begin{pmatrix} 1 & & n & 0 & -n & & & -1 \\ \bar{\sigma}_{-1,2}\lambda \dots \bar{\sigma}_{-n,2}\lambda & \bar{\sigma}_{02}\mu & 0 & 0 & \bar{\sigma}_{n-2,2} \dots \bar{\sigma}_{12} \end{pmatrix} \\
& + \begin{pmatrix} 1 & & n & 0 & -n & & & -1 \\ 0 & \dots & 0 & 0 & x_1 & 0 & \dots & 0 & x_2 & 0 \end{pmatrix}
\end{aligned}$$

for some $x_1, x_2 \in R$. Its last entry clearly does not lie in I . Hence $[\sigma, g_1] \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Clearly $[\sigma, g_1]_{n-1,*} = e_{n-1}^t$ and $[\sigma, g_1]_{*,-(n-1)} = e_{-(n-1)}$. Set $f_2 := P_{1,n-1}$ and $\tau := f_2[\sigma, g_1]$. Then $\tau_{1*} = e_1^t$ and $\tau_{*, -1} = e_{-1}$. Since $U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ is normal, $\tau \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. One can proceed now as in case 1.1.1.

case 1.2 Assume that $A \equiv e(\text{mod } I) \wedge v \equiv 0(\text{mod } \tilde{I}) \wedge C \equiv 0(\text{mod } I) \wedge D \equiv e(\text{mod } I)$ and $B \not\equiv 0(\text{mod } I)$. Recall that

$$\sigma = \left(\begin{array}{c|c|c} A & t & B \\ \hline v & z & w \\ \hline C & u & D \end{array} \right) = \left(\begin{array}{cc|c|cc} A_1 & A_2 & t_1 & B_1 & B_2 \\ 0 & e^{2 \times 2} & 0 & B_3 & B_4 \\ \hline v & & z & & w \\ \hline C & & u & & D \end{array} \right)$$

where $A, B, C, D \in M_n(R)$, $t, u \in M_{n \times 1}(R)$, $v, w \in M_{1 \times n}(R)$, $z \in R$, $A_1, B_2 \in M_{n-2}(R)$, $A_2, B_1 \in M_{(n-2) \times 2}(R)$, $B_3 \in M_2(R)$, $B_4 \in M_{2 \times (n-2)}(R)$ and $t_1 \in M_{(n-2) \times 1}(R)$.

case 1.2.1 Assume that $B_3 \not\equiv 0(\text{mod } I) \vee B_4 \not\equiv 0(\text{mod } I)$.

Set $g_1 := T_{-n,n-1}(1)$ and $\omega := [\sigma^{-1}, g_1]$. Then

$$\begin{aligned}
& \omega \\
& = [\sigma^{-1}, g_1] \\
& = (e + \sigma'_{*, -n} \sigma_{n-1,*} - \sigma'_{*, -(n-1)} \lambda \sigma_{n*}) g_1^{-1}
\end{aligned}$$

$$\begin{aligned}
& \begin{pmatrix} 1 & \bar{\lambda}\bar{\sigma}_{n,-1} \\ & \vdots \\ n & \bar{\lambda}\bar{\sigma}_{n,-n} \\ 0 & \sigma'_{0,-n} \\ -n & 1 \\ & 0 \\ & \vdots \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & n & 0 & -n & -1 \\ 0 & \dots & 0 & 1 & 0 & 0 & \sigma_{n-1,-n} & \dots & \sigma_{n-1,-1} \end{pmatrix} \\
& = (e + \begin{pmatrix} 1 & \bar{\lambda}\bar{\sigma}_{n-1,-1} \\ & \vdots \\ n & \bar{\lambda}\bar{\sigma}_{n-1,-n} \\ 0 & \sigma'_{0,-(n-1)} \\ -n & 0 \\ & 1 \\ & 0 \\ & \vdots \\ -1 & 0 \end{pmatrix} \lambda \begin{pmatrix} 1 & n & 0 & -n & -1 \\ 0 & \dots & 0 & 1 & 0 & \sigma_{n,-n} & \dots & \sigma_{n,-1} \end{pmatrix}) g_1^{-1}
\end{aligned}$$

Since $B_3 \not\equiv 0 \pmod{I}$ or $B_4 \not\equiv 0 \pmod{I}$, $(\omega_{-n,-n}, \dots, \omega_{-n,-1}) \not\equiv (1, 0, \dots, 0) \pmod{I}$ or $(\omega_{-(n-1),-n}, \dots, \omega_{-(n-1),-1}) \not\equiv (0, 1, 0, \dots, 0) \pmod{I}$. Hence $\omega \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Further $\omega_{-2,*} = e_{-2}^t$ and $\omega_{*2} = e_2$. Set $f_2 := P_{1,-2}$ and $\tau := f_2 \omega$. Then $\tau_{1*} = e_1^t$ and $\tau_{*, -1} = e_{-1}$. Since $U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ is normal, $\tau \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. One can proceed now as in case 1.1.1.

case 1.2.2 Assume that $B_3 \equiv 0 \pmod{I} \wedge B_4 \equiv 0 \pmod{I}$ and $B_2 \not\equiv 0 \pmod{I}$.

Set

$$\xi := \prod_{k=-(n-2)}^{-1} T_{n-1,k}(-\sigma_{n-1,k}) \prod_{k=-(n-2)}^{-1} T_{nk}(-\sigma_{nk}) \in EU_{2n+1}(I, \Omega) \subseteq H$$

and $\omega := \sigma \xi$. Then ω has the form

$$\omega = \left(\begin{array}{c|c|c} A' & t' & B' \\ \hline v' & z' & w' \\ \hline C' & u' & D' \end{array} \right) = \left(\begin{array}{c|c|c} A'_1 & A'_2 & t'_1 \\ 0 & e^{2 \times 2} & 0 \\ \hline v' & z' & w' \\ \hline C' & u' & D' \end{array} \begin{array}{c|c} B'_1 & B'_2 \\ \hline B'_3 & 0 \end{array} \right)$$

where A' has the same size as A , B' has the same size as B and so on. Clearly $\omega \equiv \sigma \pmod{I, \tilde{I}, I_0, \tilde{I}_0}$ and hence $B'_2 \not\equiv 0 \pmod{I}$. Therefore there are an $i \in \{1, \dots, n-2\}$ and a $j \in \{-(n-2), \dots, -1\}$ such that $\omega_{ij} \notin I$. Choose a $k \in \{1, \dots, n-2\} \setminus \{-j\}$, set $f_{11} := T_{jk}(-1)$ and $\rho := f_{11} \omega$. Then ρ has the form

$$\rho = \left(\begin{array}{c|c|c} A'' & t'' & B'' \\ \hline v'' & z'' & w'' \\ \hline C'' & u'' & D'' \end{array} \right) = \left(\begin{array}{c|c|c} A''_1 & A''_2 & t''_1 \\ 0 & e^{2 \times 2} & 0 \\ \hline v'' & z'' & w'' \\ \hline C'' & u'' & D'' \end{array} \begin{array}{c|c} B''_1 & B''_2 \\ \hline B''_3 & 0 \end{array} \right)$$

where A'' has the same size as A' , B'' has the same size as B' and so on. Further $\rho_{ik} \not\equiv \delta_{ik} \pmod{I}$. Hence $A'' \not\equiv e \pmod{I}$ and thus one can proceed as in case 1.1.

case 1.2.3 Assume that $B_3 \equiv 0(\text{mod } I) \wedge B_4 \equiv 0(\text{mod } I) \wedge B_2 \equiv 0(\text{mod } I)$.

Since $B \not\equiv 0(\text{mod } I)$, it follows that $B_1 \not\equiv 0(\text{mod } I)$. Hence there are an $i \in \{1, \dots, n-2\}$ and a $j \in \{-n, -(n-1)\}$ such that $\omega_{ij} \notin I$. Set $f_{11} := T_{j,-1}(-1)$ and $\omega := f_{11}\sigma$. Then ω has the form

$$\omega = \left(\begin{array}{c|c|c} A' & t' & B' \\ \hline v' & z' & w' \\ \hline C' & u' & D' \end{array} \right) = \left(\begin{array}{cc|c|cc} A'_1 & A'_2 & t'_1 & B'_1 & B'_2 \\ 0 & e^{2 \times 2} & 0 & B'_3 & B'_4 \\ \hline v' & z' & w' & & \\ \hline C' & u' & D' & & \end{array} \right)$$

where A' has the same size as A , B' has the same size as B and so on. Clearly $A' \equiv e(\text{mod } I) \wedge v' \equiv 0(\text{mod } I) \wedge C' \equiv 0(\text{mod } I) \wedge D' \equiv e(\text{mod } I) \wedge B'_3 \equiv 0(\text{mod } I) \wedge B'_4 \equiv 0(\text{mod } I)$. Further $\omega_{i,-1} \notin I$ and hence $B'_2 \not\equiv 0(\text{mod } I)$. Thus one can proceed as in case 1.2.2.

case 1.3 Assume that $A \equiv e(\text{mod } I) \wedge v \equiv 0(\text{mod } \tilde{I}) \wedge C \equiv 0(\text{mod } I) \wedge D \equiv e(\text{mod } I) \wedge B \equiv 0(\text{mod } I)$ and $w \not\equiv 0(\text{mod } \tilde{I})$.

Set

$$\xi := \prod_{k=-(n-2)}^{-1} T_{n-1,k}(-\sigma_{n-1,k}) \prod_{k=-(n-2)}^{-1} T_{nk}(-\sigma_{nk}) \in EU_{2n+1}(I, \Omega) \subseteq H$$

and $\omega := \sigma\xi$. Then ω has the form

$$\omega = \left(\begin{array}{c|c|c} A' & t' & B' \\ \hline v' & z' & w' \\ \hline C' & u' & D' \end{array} \right) = \left(\begin{array}{cc|c|cc} A'_1 & A'_2 & t'_1 & B'_1 & B'_2 \\ 0 & e^{2 \times 2} & 0 & B'_3 & 0 \\ \hline v' & z' & w' & & \\ \hline C' & u' & D' & & \end{array} \right)$$

where A' has the same size as A , B' has the same size as B and so on. Clearly $\omega \equiv \sigma(\text{mod } I, \tilde{I}, I_0, \tilde{I}_0)$ and hence $w' \not\equiv 0(\text{mod } \tilde{I})$. Therefore there is a $j \in \Theta_-$ such that $w'_j \notin \tilde{I}$. By the definition of \tilde{I} there is an $a \in J(\Delta)$ such that $\bar{a}\mu w'_j \notin I$. Choose a $b \in R$ such that $(a, b) \in \Delta$ and set $f_{11} := T_{-1}(a, b)$ and $\rho := f_{11}\omega$. Then ρ has the form

$$\rho = \left(\begin{array}{c|c|c} A'' & t'' & B'' \\ \hline v'' & z'' & w'' \\ \hline C'' & u'' & D'' \end{array} \right) = \left(\begin{array}{cc|c|cc} A''_1 & A''_2 & t''_1 & B''_1 & B''_2 \\ 0 & e^{2 \times 2} & 0 & B''_3 & 0 \\ \hline v'' & z'' & w'' & & \\ \hline C'' & u'' & D'' & & \end{array} \right)$$

where A'' has the same size as A' , B'' has the same size as B' and so on. Further $\rho_{-1,j} \not\equiv \delta_{-1,j}(\text{mod } I)$. Hence $D'' \not\equiv e(\text{mod } I)$ and thus one can proceed as in case 1.1.

case 1.4 Assume that $A \equiv e(\text{mod } I) \wedge v \equiv 0(\text{mod } \tilde{I}) \wedge C \equiv 0(\text{mod } I) \wedge D \equiv e(\text{mod } I) \wedge B \equiv 0(\text{mod } I) \wedge w \equiv 0(\text{mod } \tilde{I})$.

Clearly $z \not\equiv 1(\text{mod } \tilde{I}_0)$ since $\sigma \notin U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Set

$$\xi := \prod_{k=-(n-2)}^{-1} T_{n-1,k}(-\sigma_{n-1,k}) \prod_{k=-(n-2)}^{-1} T_{nk}(-\sigma_{nk}) \in U$$

and $\omega := \sigma\xi$. Then ω has the form

$$\omega = \left(\begin{array}{c|c|c} A' & t' & B' \\ \hline v' & z' & w' \\ \hline C' & u' & D' \end{array} \right) = \left(\begin{array}{cc|c|cc} A'_1 & A'_2 & t'_1 & B'_1 & B'_2 \\ 0 & e^{2 \times 2} & 0 & B'_3 & 0 \\ \hline v' & z' & w' & & \\ \hline C' & u' & D' & & \end{array} \right)$$

where A' has the same size as A , B' has the same size as B and so on. Clearly $\omega \equiv \sigma(\text{mod } I, \tilde{I}, I_0, \tilde{I}_0)$ and hence $z' \not\equiv 1(\text{mod } \tilde{I}_0)$. It follows from the definition of \tilde{I}_0 that there is an $a \in J(\Delta)$ such that $(z' - 1)a \notin \tilde{I}$.

Choose a $b \in R$ such that $(a, b) \in \Delta$ and set $f_{11} := T_{-1}(a, b)$ and $\rho := f_{11}\omega$. Then ρ has the form

$$\rho = \left(\begin{array}{c|c|c} A'' & t'' & B'' \\ \hline v'' & z'' & w'' \\ \hline C'' & u'' & D'' \end{array} \right) = \left(\begin{array}{cc|c|cc} A_1'' & A_2'' & t_1'' & B_1'' & B_2'' \\ 0 & e^{2 \times 2} & 0 & B_3'' & 0 \\ \hline v'' & z'' & w'' & & \\ \hline C'' & u'' & D'' & & \end{array} \right)$$

where A'' has the same size as A' , B'' has the same size as B' and so on. Further $\rho_{01} \notin \tilde{I}$. Hence $v'' \not\equiv 0 \pmod{\tilde{I}}$ and thus one can proceed as in case 1.1.

case 2 Assume that (2) in Lemma 42 holds.

Replace h by $[h, T_k(y, z)]$ in case 1.

Part II Assume that $h \notin CU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ and $n = 3$.

By Lemma 42, either (1) or (2) in Lemma 42 holds.

case 1 Assume that (1) in Lemma 42 holds.

Then there are $f_0, f_1 \in E$ and $g_0 \in \text{Gen}(E)$ such that $\sigma := f_1[f_0 h, g_0] \notin U_7((R, \Delta), (I, \Omega_{max}^I))$ and σ has the form

$$\sigma = \left(\begin{array}{c|c|c} A & t & B \\ \hline v & z & w \\ \hline C & u & D \end{array} \right) = \left(\begin{array}{ccc|c|c} * & * & * & * & \\ * & * & * & * & B \\ 0 & 0 & 1 & 0 & \\ \hline v & z & w & & \\ \hline C & u & D & & \end{array} \right)$$

where $A, B, C, D \in M_3(R)$, $t, u \in M_{3 \times 1}(R)$, $v, w \in M_{1 \times 3}(R)$ and $z \in R$ (see Part I).

case 1.1 Assume that $\sigma_{01} \notin \tilde{I} \vee \sigma_{-3,1} \notin I \vee \sigma_{-2,1} \notin I \vee \sigma_{-1,1} \notin I \vee \sigma_{02} \notin \tilde{I} \vee \sigma_{-3,2} \notin I \vee \sigma_{-2,2} \notin I \vee \sigma_{-1,2} \notin I$. Set $g_1 := T_{1,-2}(1)$ and $\omega := [\sigma, g_1]$. Then

$$\begin{aligned} & \omega \\ &= [\sigma, g_1] \\ &= (e + \sigma_{*1}\sigma'_{-2,*} - \sigma_{*2}\bar{\lambda}\sigma'_{-1,*})g_1^{-1} \\ &= (e + \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ 0 \\ \sigma_{01} \\ \sigma_{-3,1} \\ \sigma_{-2,1} \\ \sigma_{-1,1} \end{pmatrix} \begin{pmatrix} \bar{\sigma}_{-1,2}\lambda & \bar{\sigma}_{-2,2}\lambda & \bar{\sigma}_{-3,2}\lambda & \bar{\sigma}_{02}\mu & 0 & \bar{\sigma}_{22} & \bar{\sigma}_{12} \end{pmatrix} \\ & \quad - \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \\ 0 \\ \sigma_{02} \\ \sigma_{-3,2} \\ \sigma_{-2,2} \\ \sigma_{-1,2} \end{pmatrix} \bar{\lambda} \begin{pmatrix} \bar{\sigma}_{-1,1}\lambda & \bar{\sigma}_{-2,1}\lambda & \bar{\sigma}_{-3,1}\lambda & \bar{\sigma}_{01}\mu & 0 & \bar{\sigma}_{21} & \bar{\sigma}_{11} \end{pmatrix})g_1^{-1}. \end{aligned}$$

Assume that $\omega \in U_7((R, \Delta), (I, \Omega_{max}^I))$. Then $\omega \equiv e \pmod{I, \tilde{I}, I_0, \tilde{I}_0}$ by Remark 29. Hence $\sigma_{*1}\bar{\sigma}_{-j,2}\lambda - \sigma_{*2}\bar{\lambda}\bar{\sigma}_{-j,1}\lambda = \omega_{*j} - e_j \equiv 0 \pmod{I, \tilde{I}} \forall j \in \{1, 2, 3\}$ and $\sigma_{*1}\bar{\sigma}_{02}\mu - \sigma_{*2}\bar{\lambda}\bar{\sigma}_{01}\mu = \omega_{*0} - e_0 \equiv 0 \pmod{I_0, \tilde{I}_0}$. By multiplying σ'_{1*} resp. σ'_{2*} from the left we get $\sigma_{-3,1}, \sigma_{-2,1}, \sigma_{-1,1}, \sigma_{-3,2}, \sigma_{-2,2}, \sigma_{-1,2} \in I$ and $\sigma_{01}, \sigma_{02} \in \tilde{I}$ (note that $\sigma'_{10}, \sigma'_{20} \in J(\Delta)\mu$ by Lemma 20). Since this is a contradiction, $\omega \notin U_7((R, \Delta), (I, \Omega_{max}^I))$.

Further $\omega_{3*} = e_3^t$ and $\omega_{*, -3} = e_{-3}$. Set $f_2 := P_{13}$ and $\tau := f_2 \omega$. Then $\tau_{1*} = e_1^t$ and $\tau_{*, -1} = e_{-1}$. Since $U_7((R, \Delta), (I, \Omega_{max}^I))$ is normal, $\tau \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. One can proceed now as in Part I, case 1.1.1.

case 1.2 Assume that $\sigma_{01}, \sigma_{02} \in \tilde{I}$, $\sigma_{-3,1}, \sigma_{-2,1}, \sigma_{-1,1}, \sigma_{-3,2}, \sigma_{-2,2}, \sigma_{-1,2} \in I$ and $\sigma_{3,-3} \notin I \vee \sigma_{3,-2} \notin I$. By [1, chapter III, Lemma 4.3] there are $x_1, x_2 \in I$ such that $\sigma_{-1,1} + x_2(x_1\sigma_{-1,1} + \sigma_{-2,1}) \in \text{rad}(R)$ where $\text{rad}(R)$ denotes the Jacobson radical of R . Set $\xi_0 := T_{-1,-2}(x_2)T_{-2,-1}(x_1) \in EU_{2n+1}(I, \Omega) \subseteq H$. Then $\rho := \xi_0 \sigma$ has the form

$$\rho = \left(\begin{array}{c|c|c} A' & t' & B' \\ \hline v' & z' & w' \\ \hline C' & u' & D' \end{array} \right) = \left(\begin{array}{ccc|c|c} * & * & * & * & \\ * & * & * & * & B' \\ 0 & 0 & 1 & 0 & \\ \hline & v' & & z' & w' \\ \hline & C' & & u' & D' \end{array} \right)$$

where A has the same size as A , B' has the same size as B and so on. Further $\rho \equiv \sigma \pmod{I, I_0, \tilde{I}, \tilde{I}_0}$ and $\rho_{-1,1} = \sigma_{-1,1} + x_2(x_1\sigma_{-1,1} + \sigma_{-2,1}) \in \text{rad}(R)$. Set $g_1 := T_{13}(1)$ and $\omega := [\rho^{-1}, g_1]$. Then

$$\begin{aligned} & \omega \\ &= [\rho^{-1}, g_1] \\ &= (e + \rho'_{*1}\rho_{3*} - \rho'_{*, -3}\rho_{-1,*})g_1^{-1} \\ &= (e + \begin{pmatrix} \bar{\lambda}\bar{\rho}_{-1,-1}\lambda \\ \bar{\lambda}\bar{\rho}_{-1,-2}\lambda \\ \bar{\lambda}\bar{\rho}_{-1,-3}\lambda \\ \rho'_{01} \\ \bar{\rho}_{-1,3}\lambda \\ \bar{\rho}_{-1,2}\lambda \\ \bar{\rho}_{-1,1}\lambda \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & \rho_{3,-3} & \rho_{3,-2} & \rho_{3,-1} \end{pmatrix} \\ & \quad - \begin{pmatrix} \bar{\lambda}\bar{\rho}_{3,-1} \\ \bar{\lambda}\bar{\rho}_{3,-2} \\ \bar{\lambda}\bar{\rho}_{3,-3} \\ \rho'_{0,-3} \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \rho_{-1,1} & \rho_{-1,2} & \rho_{-1,3} & \rho_{-1,0} & \rho_{-1,-3} & \rho_{-1,-2} & \rho_{-1,-1} \end{pmatrix})g_1^{-1}. \end{aligned}$$

Assume that $\omega \in U_7((R, \Delta), (I, \Omega_{max}^I))$. Then $\rho'_{*1}\rho_{3j} - \rho'_{*, -3}\rho_{-1,j} = \omega_{*j} - e_j \equiv 0 \pmod{I, \tilde{I}} \forall j \in \{-3, -2\}$. By multiplying ρ_{1*} from the left we get that $\rho_{3,-3}, \rho_{3,-2} \in I$. Since that is a contradiction, $\omega \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. Obviously $\omega_{-1,*} \equiv e_{-1}^t \pmod{I, I_0}$ and $\omega_{-1,-1} \equiv 1 \pmod{\text{rad}(R)}$ (note that $\text{rad}(R)$ is involution invariant since $^-$ defines a bijection between maximal left and maximal right ideals of R). Set $f_2 := P_{3,-1}$ and $\zeta := f_2 \omega$. Then $\zeta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. Further $\zeta_{3*} \equiv e_3^t \pmod{I, I_0}$ and $\zeta_{33} \equiv 1 \pmod{\text{rad}(R)}$. By Nakayama's lemma ζ_{33} is invertible. Set $\xi_2 := T_{32}(-(\zeta_{33})^{-1}\zeta_{32})T_{31}(-(\zeta_{33})^{-1}\zeta_{31})T_{3,-1}(-(\zeta_{33})^{-1}\zeta_{3,-1})T_{3,-2}(-(\zeta_{33})^{-1}\zeta_{3,-2}) \subseteq EU_{2n+1}(I, \Omega) \subseteq H$ and $\eta := \xi_2 \zeta$. Then η has the form

$$\eta = \left(\begin{array}{c|c|c} A'' & t'' & B'' \\ \hline v'' & z'' & w'' \\ \hline C'' & u'' & D'' \end{array} \right) = \left(\begin{array}{ccc|c|ccc} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ 0 & 0 & \eta_{33} & \eta_{30} & \eta_{3,-3} & 0 & 0 \\ \hline & v'' & & z'' & & & w'' \\ \hline & C'' & & u'' & & & D'' \end{array} \right)$$

where A'' has the same size as A' , B'' has the same size as B' and so on. Since $\zeta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$, $\eta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. Further $\eta_{33} \equiv 1 \pmod{\text{rad}(R) \cap I}$, $\eta_{30} \in I_0$ and $\eta_{3,-3} \in I$. Since $\eta_{3*} \equiv e_3^t \pmod{I, I_0}$, $\eta_{*, -3} \equiv e_{-3} \pmod{I, \tilde{I}}$ (follows from Lemma 20). Hence $\eta_{1,-3}, \eta_{2,-3}, \eta_{-2,-3}, \eta_{-1,-3} \in I$, $\eta_{-3,-3} \equiv 1 \pmod{I}$ and $\eta_{0,-3} \in \tilde{I}$.

case 1.2.1 Assume that $\eta_{01} \notin \tilde{I} \vee \eta_{-3,1} \notin I \vee \eta_{-2,1} \notin I \vee \eta_{-1,1} \notin I \vee \eta_{02} \notin \tilde{I} \vee \eta_{-3,2} \notin I \vee \eta_{-2,2} \notin I \vee \eta_{-1,2} \notin I$. See case 1.1.

case 1.2.2 Assume that $\eta_{01}, \eta_{02} \in \tilde{I}$, $\eta_{-3,1}, \eta_{-2,1}, \eta_{-1,1}, \eta_{-3,2}, \eta_{-2,2}, \eta_{-1,2} \in I$ and $\eta_{1,-2} \notin I \vee \eta_{2,-2} \notin I \vee \eta_{0,-2} \notin \tilde{I} \vee \eta_{-3,-2} \notin I \vee \eta_{1,-1} \notin I \vee \eta_{2,-1} \notin I \vee \eta_{0,-1} \notin \tilde{I} \vee \eta_{-3,-1} \notin I$. Set $g_2 := T_{-1,2}(1)$ and $\theta := [\eta, g_2]$. Then

$$\begin{aligned} & \theta \\ &= [\eta, g_2] \\ &= (e + \eta_{*, -1} \eta'_{2*} - \eta_{*, -2} \lambda \eta'_{1*}) g_2^{-1} \\ &= (e + \begin{pmatrix} \eta_{1,-1} \\ \eta_{2,-1} \\ 0 \\ \eta_{0,-1} \\ \eta_{-3,-1} \\ \eta_{-2,-1} \\ \eta_{-1,-1} \end{pmatrix} \bar{\lambda} \begin{pmatrix} \bar{\eta}_{-1,-2} \lambda & \bar{\eta}_{-2,-2} \lambda & \bar{\eta}_{-3,-2} \lambda & \bar{\eta}_{0,-2} \mu & 0 & \bar{\eta}_{2,-2} & \bar{\eta}_{1,-2} \end{pmatrix} \\ & \quad - \begin{pmatrix} \eta_{1,-2} \\ \eta_{2,-2} \\ 0 \\ \eta_{0,-2} \\ \eta_{-3,-2} \\ \eta_{-2,-2} \\ \eta_{-1,-2} \end{pmatrix} \lambda \bar{\lambda} \begin{pmatrix} \bar{\eta}_{-1,-1} \lambda & \bar{\eta}_{-2,-1} \lambda & \bar{\eta}_{-3,-1} \lambda & \bar{\eta}_{0,-1} \mu & 0 & \bar{\eta}_{2,-1} & \bar{\eta}_{1,-1} \end{pmatrix}) g_2^{-1}. \end{aligned}$$

Assume that $\theta \in U_7((R, \Delta), (I, \Omega_{max}^I))$. Then $\eta_{*, -1} \bar{\lambda} \bar{\eta}_{-j, -2} \lambda^{(\epsilon(j)+1)/2} - \eta_{*, -2} \bar{\eta}_{-j, -1} \lambda^{(\epsilon(j)+1)/2} = \theta_{*j} - e_j \equiv 0 \pmod{I, \tilde{I}} \forall j \in \{3, -2, -1\}$ and $\eta_{*, -1} \bar{\lambda} \bar{\eta}_{0, -2} \mu - \eta_{*, -2} \bar{\eta}_{0, -1} \mu = \theta_{*0} - e_0 \equiv 0 \pmod{I_0, \tilde{I}_0}$. It follows that $\eta_{1,-2}, \eta_{2,-2}, \eta_{-3,-2}, \eta_{1,-1}, \eta_{2,-1}, \eta_{-3,-1} \in I$ and $\eta_{0,-2}, \eta_{0,-1} \in \tilde{I}$ (multiply $\eta'_{-1,*}$ resp. $\eta'_{-2,*}$ from the left). Since that is a contradiction, $\theta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. Clearly $\theta_{3*} = e_3^t$ and $\theta_{*, -3} = e_{-3}$. Set $f_3 := P_{13}$. Then $(f_3 \theta)_{1*} = e_1^t$ and $(f_3 \theta)_{*, -1} = e_{-1}$. Since $f_3 \theta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$, one can proceed as in Part I, case 1.1.1.

case 1.2.3 Assume that $\eta_{01}, \eta_{02}, \eta_{0,-2}, \eta_{0,-1} \in \tilde{I}$, $\eta_{-3,1}, \eta_{-2,1}, \eta_{-1,1}, \eta_{-3,2}, \eta_{-2,2}, \eta_{-1,2}, \eta_{1,-2}, \eta_{2,-2}, \eta_{-3,-2}, \eta_{1,-1}, \eta_{2,-1}, \eta_{-3,-1} \in I$ and $\eta_{11} - 1 \notin I \vee \eta_{21} \notin I \vee \eta_{-1,3} \notin I \vee \eta_{-1,-2} \notin I \vee \eta_{-1,-1} - 1 \notin I$. Set $g_2 := T_{13}(1)$ and $\theta := [\eta^{-1}, g_2]$. Then

$$\begin{aligned} & \theta \\ &= [\eta^{-1}, g_2] \\ &= (e + \eta'_{*1} \eta_{3*} - \eta'_{*, -3} \eta_{-1,*}) g_2^{-1} \\ &= (e + \begin{pmatrix} \bar{\lambda} \bar{\eta}_{-1,-1} \lambda \\ \bar{\lambda} \bar{\eta}_{-1,-2} \lambda \\ \bar{\lambda} \bar{\eta}_{-1,-3} \lambda \\ \eta'_{01} \\ \bar{\eta}_{-1,3} \lambda \\ \bar{\eta}_{-1,2} \lambda \\ \bar{\eta}_{-1,1} \lambda \end{pmatrix} \begin{pmatrix} 0 & 0 & \eta_{33} & \eta_{30} & \eta_{3,-3} & 0 & 0 \end{pmatrix}) g_2^{-1} \end{aligned}$$

$$- \begin{pmatrix} 0 \\ 0 \\ \bar{\lambda}\bar{\eta}_{3,-3} \\ \eta'_{0,-3} \\ \bar{\eta}_{33} \\ 0 \\ 0 \end{pmatrix} (\eta_{-1,1} \quad \eta_{-1,2} \quad \eta_{-1,3} \quad \eta_{-1,0} \quad \eta_{-1,-3} \quad \eta_{-1,-2} \quad \eta_{-1,-1}) g_2^{-1}.$$

Assume that $\theta \in U_7((R, \Delta), (I, \Omega_{max}^I))$. Then $\bar{\lambda}\bar{\eta}_{-1,-1}\lambda\eta_{33} - 1 = \theta_{13} \in I$ and $\bar{\lambda}\bar{\eta}_{-1,-2}\lambda\eta_{33} = \theta_{23} \in I$. Since $\eta_{33} \equiv 1 \pmod{I}$, it follows that $\eta_{-1,-1} - 1, \eta_{-1,-2} \in I$. Since by assumption $\theta \in U_7((R, \Delta), (I, \Omega_{max}^I))$,

$$\begin{aligned} & \theta_{*3} - e_3 \\ &= \eta'_{*1}\eta_{33} - \eta'_{*, -3}\eta_{-1,3} - (1 \quad 0 \quad -\bar{\lambda}\bar{\eta}_{3,-3}\eta_{-1,1} \quad -\eta'_{0,-3}\eta_{-1,1} \quad -\bar{\eta}_{33}\eta_{-1,1} \quad 0 \quad 0)^t \end{aligned}$$

is congruent to 0 modulo I, \tilde{I} . By multiplying $\eta_{-3,*}$ from the left we get that $\eta_{-1,3} \in I$ since $\eta_{-3,1}, \eta_{-1,1} \in I$. Further $\bar{\eta}_{-1,3}\lambda\eta_{30} - \bar{\eta}_{33}\eta_{-1,0} = \theta_{-3,0} \in I_0$. Since $\eta_{-1,3} \in I$ and $\eta_{33} \equiv 1 \pmod{I}$, it follows that $\eta_{-1,0} \in I_0$ (note that $I \subseteq I_0$). Hence $\eta_{-1,*} \equiv e_{-1}^t \pmod{I, I_0}$ and therefore $\eta_{*1} \equiv e_1 \pmod{I, \tilde{I}}$ (follows from Lemma 20). This implies $\eta_{11} - 1, \eta_{21} \in I$. Since that is a contradiction, $\theta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. One checks easily that θ has the form

$$\theta = \left(\begin{array}{ccc|ccc} 1 & 0 & * & + & + & 0 & + \\ 0 & 1 & * & + & + & 0 & + \\ + & + & + & + & + & + & + \\ \hline + & + & * & + & + & + & + \\ \hline + & + & * & * & + & * & * \\ 0 & 0 & + & + & + & 1 & + \\ 0 & 0 & + & + & + & 0 & + \end{array} \right)$$

where a $+$ at a position (i, j) means that the entry at this position is congruent to δ_{ij} modulo I if $i, j \in \Theta_{hb}$, congruent to δ_{0j} modulo \tilde{I} if $i = 0, j \in \Theta_{hb}$, congruent to δ_{i0} modulo I_0 if $i \in \Theta_{hb}, j = 0$ and congruent to 1 modulo \tilde{I}_0 if $i = j = 0$. A $*$ stands for an arbitrary entry.

case 1.2.3.1 Assume that $\theta_{13} \notin I \vee \theta_{23} \notin I$.

Set $g_3 := T_{-2,1}(1)$ and $\tau := [\theta^{-1}, g_3]$. Then

$$\begin{aligned} & \tau \\ &= [\theta^{-1}, g_3] \\ &= (e + \theta'_{*, -2}\theta_{1*} - \theta'_{*, -1}\lambda\theta_{2*})g_3^{-1} \\ &= (e + \begin{pmatrix} \bar{\lambda}\bar{\theta}_{2,-1} \\ 0 \\ \bar{\lambda}\bar{\theta}_{2,-3} \\ \theta'_{0,-2} \\ \theta_{23} \\ \theta_{22} \\ \theta_{21} \end{pmatrix} (\theta_{11} \quad \theta_{12} \quad \theta_{13} \quad \theta_{10} \quad \theta_{1,-3} \quad 0 \quad \theta_{1,-1}) \\ & \quad - \begin{pmatrix} \bar{\lambda}\bar{\theta}_{1,-1} \\ 0 \\ \bar{\lambda}\bar{\theta}_{1,-3} \\ \theta'_{0,-1} \\ \theta_{13} \\ \theta_{12} \\ \theta_{11} \end{pmatrix} \lambda (\theta_{21} \quad \theta_{22} \quad \theta_{23} \quad \theta_{20} \quad \theta_{2,-3} \quad 0 \quad \theta_{2,-1}))g_3^{-1}. \end{aligned}$$

Assume that $\tau \in U_7((R, \Delta), (I, \Omega_{max}^I))$. Then $\tau_{*3} - e_3 = \theta'_{*, -2}\theta_{13} - \theta'_{*, -1}\lambda\theta_{23} \equiv 0 \pmod{I, \tilde{I}}$. It follows that $\theta_{13}, \theta_{23} \in I$ which is a contradiction. Hence $\tau \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. Clearly $\tau_{2*} = e_2^t$ and $\tau_{*, -2} = e_{-2}$. Set $f_4 := P_{12}$. Then $(f_4\tau)_{1*} = e_1^t$ and $(f_4\tau)_{*, -1} = e_{-1}$. Since $f_4\theta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$, one can proceed as in Part I, case 1.1.1.

case 1.2.3.2 Assume that $\theta_{13}, \theta_{23} \in I$.

It follows that $\theta_{-3, -2}, \theta_{-3, -1} \in I$ (consider $b(\theta_{*3}, \theta_{*, -2})$ and $b(\theta_{*3}, \theta_{*, -1})$). Set $\xi_3 := T_{23}(-\theta_{23})T_{2, -1}(-\theta_{2, -1}) \in EU_{2n+1}(I, \Omega) \subseteq H$ and $\chi := \theta\xi_3$. Then χ has the form

$$\chi = \left(\begin{array}{ccc|ccc} 1 & 0 & + & + & + & + \\ 0 & 1 & 0 & + & + & 0 \\ + & + & + & + & + & + \\ \hline + & + & * & + & + & + \\ + & + & * & * & + & + \\ 0 & 0 & + & + & + & + \\ 0 & 0 & + & + & + & + \end{array} \right)$$

Since $\theta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$, $\chi \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. Set $g_3 := T_{31}(1)$ and $\psi := [\chi, g_3]$. Then

$$\begin{aligned} & \psi \\ &= [\chi, g_3] \\ &= (e + \chi_{*3}\chi'_{1*} - \chi_{*, -1}\chi'_{-3, *})g_3^{-1} \\ &= (e + \begin{pmatrix} \chi_{13} \\ 0 \\ \chi_{33} \\ \chi_{03} \\ \chi_{-3, 3} \\ \chi_{-2, 3} \\ \chi_{-1, 3} \end{pmatrix} \bar{\lambda} \begin{pmatrix} \bar{\chi}_{-1, -1}\lambda & \bar{\chi}_{-2, -1}\lambda & \bar{\chi}_{-3, -1}\lambda & \bar{\chi}_{0, -1}\mu & \bar{\chi}_{3, -1} & 0 & \bar{\chi}_{1, -1} \end{pmatrix} \\ & \quad - \begin{pmatrix} \chi_{1, -1} \\ 0 \\ \chi_{3, -1} \\ \chi_{0, -1} \\ \chi_{-3, -1} \\ \chi_{-2, -1} \\ \chi_{-1, -1} \end{pmatrix} \begin{pmatrix} \bar{\chi}_{-1, 3}\lambda & \bar{\chi}_{-2, 3}\lambda & \bar{\chi}_{-3, 3}\lambda & \bar{\chi}_{03}\mu & \bar{\chi}_{33} & 0 & \bar{\chi}_{13} \end{pmatrix})g_3^{-1}. \end{aligned}$$

Assume that $\psi \in U_7((R, \Delta), (I, \Omega_{max}^I))$. Then

$$\chi_{-3, 3}\bar{\lambda}\bar{\chi}_{-1, -1}\lambda - \chi_{-3, -1}\bar{\chi}_{-1, 3}\lambda - (\chi_{-3, 3}\bar{\lambda}\bar{\chi}_{-3, -1}\lambda - \chi_{-3, -1}\bar{\chi}_{-3, 3}\lambda) = \psi_{-3, 1} \in I.$$

It follows that $\chi_{-3, 3} \in I$ since $\chi_{-1, -1} \equiv 1 \pmod{I}$ and $\chi_{-3, -1} \in I$. Further

$$\chi_{03}\bar{\lambda}\bar{\chi}_{-1, -1}\lambda - \chi_{0, -1}\bar{\chi}_{-1, 3}\lambda - (\chi_{03}\bar{\lambda}\bar{\chi}_{-3, -1}\lambda - \chi_{0, -1}\bar{\chi}_{-3, 3}\lambda) = \psi_{01} \in \tilde{I}.$$

It follows that $\chi_{03} \in \tilde{I}$ since $\chi_{-1, -1} \equiv 1 \pmod{I}$ and $\chi_{-1, 3}, \chi_{-3, -1}, \chi_{-3, 3} \in I$. Hence $\chi \in U_7((R, \Delta), (I, \Omega_{max}^I))$ which is a contradiction. Therefore $\psi \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. Clearly $\psi_{2*} = e_2^t$ and $\psi_{*, -2} = e_{-2}$. Set $f_4 := P_{12}$. Then $(f_4\psi)_{1*} = e_1^t$ and $(f_4\psi)_{*, -1} = e_{-1}$. Since $f_4\psi \notin U_7((R, \Delta), (I, \Omega_{max}^I))$, one can proceed as in Part I, case 1.1.1.

case 1.2.4 Assume that $\eta_{01}, \eta_{02}, \eta_{0, -2}, \eta_{0, -1} \in \tilde{I}$, $\eta_{-3, 1}, \eta_{-2, 1}, \eta_{-1, 1}, \eta_{-3, 2}, \eta_{-2, 2}, \eta_{-1, 2}, \eta_{1, -2}, \eta_{2, -2}, \eta_{-3, -2}, \eta_{1, -1}, \eta_{2, -1}, \eta_{-3, -1}, \eta_{11} - 1, \eta_{21}, \eta_{-1, 3}, \eta_{-1, -2}, \eta_{-1, -1} - 1 \in I$ and $\eta_{12} \notin I \vee \eta_{22} - 1 \notin I \vee \eta_{-2, 3} \notin I \vee \eta_{-2, -2} - 1 \notin I$

$I \vee \eta_{-2,-1} \notin I \vee \eta_{13} \notin I \vee \eta_{23} \notin I$.

Set $g_2 := T_{21}(1)$ and $\theta := [\eta, g_2]$. Then

$$\begin{aligned} & \theta \\ &= [\eta, g_2] \\ &= (e + \eta_{*2}\eta'_{1*} - \eta_{*, -1}\eta'_{-2,*})g_2^{-1} \\ &= (e + \begin{pmatrix} \eta_{12} \\ \eta_{22} \\ 0 \\ \eta_{02} \\ \eta_{-3,2} \\ \eta_{-2,2} \\ \eta_{-1,2} \end{pmatrix} \bar{\lambda} \begin{pmatrix} \bar{\eta}_{-1,-1}\lambda & \bar{\eta}_{-2,-1}\lambda & \bar{\eta}_{-3,-1}\lambda & \bar{\eta}_{0,-1}\mu & 0 & \bar{\eta}_{2,-1} & \bar{\eta}_{1,-1} \end{pmatrix} \\ & \quad - \begin{pmatrix} \eta_{1,-1} \\ \eta_{2,-1} \\ 0 \\ \eta_{0,-1} \\ \eta_{-3,-1} \\ \eta_{-2,-1} \\ \eta_{-1,-1} \end{pmatrix} \begin{pmatrix} \bar{\eta}_{-1,2}\lambda & \bar{\eta}_{-2,2}\lambda & \bar{\eta}_{-3,2}\lambda & \bar{\eta}_{02}\mu & 0 & \bar{\eta}_{22} & \bar{\eta}_{12} \end{pmatrix})g_2^{-1}. \end{aligned}$$

Assume that $\theta \in U_7((R, \Delta), (I, \Omega_{max}^I))$. Then $\eta_{*2}\bar{\lambda}\bar{\eta}_{1,-1} - \eta_{*, -1}\bar{\eta}_{12} = \theta_{*, -1} - e_{-1} \equiv 0 \pmod{I, \tilde{I}}$. It follows that $\eta_{12} \in I$. Further $\eta_{*2}\bar{\lambda}\bar{\eta}_{2,-1} - \eta_{*, -1}\bar{\eta}_{22} + e_{-1} \equiv \theta_{*, -1} - e_{-1} \equiv 0 \pmod{I, \tilde{I}}$. By multiplying $\eta'_{-1,*}$ from the left we get that $-\bar{\eta}_{22} + \bar{\eta}_{11} \in I$ which implies $\eta_{22} \equiv 1 \pmod{I}$ since $\eta_{11} \equiv 1 \pmod{I}$. Hence $\eta_{*2} \equiv e_2 \pmod{I, \tilde{I}}$ and therefore $\eta_{-2,*} \equiv e_{-2}^t \pmod{I, I_0}$. This implies $\eta_{*, -2} \equiv e_{-2} \pmod{I, \tilde{I}}$ and $\eta_{*, -1} \equiv e_{-1} \pmod{I, \tilde{I}}$ and therefore $\eta_{2*} \equiv e_2^t \pmod{I, I_0}$ and $\eta_{1*} \equiv e_1^t \pmod{I, I_0}$. Thus $\eta_{-2,3}, \eta_{-2,-2} - 1, \eta_{-2,-1}, \eta_{13}, \eta_{23} \in I$. Since that is a contradiction, $\theta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. Clearly $\theta_{3*} = e_3^t$ and $\theta_{*, -3} = e_{-3}$. Set $f_3 := P_{13}$. Then $(f_3\theta)_{1*} = e_1^t$ and $(f_3\theta)_{*, -1} = e_{-1}$. Since $f_3\theta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$, one can proceed as in Part I, case 1.1.1.

case 1.2.5 Assume that $\eta_{01}, \eta_{02}, \eta_{0,-2}, \eta_{0,-1} \in \tilde{I}$, $\eta_{-3,1}, \eta_{-2,1}, \eta_{-1,1}, \eta_{-3,2}, \eta_{-2,2}, \eta_{-1,2}$, $\eta_{1,-2}, \eta_{2,-2}, \eta_{-3,-2}$, $\eta_{1,-1}, \eta_{2,-1}, \eta_{-3,-1}, \eta_{11} - 1, \eta_{21}, \eta_{-1,3}, \eta_{-1,-2}, \eta_{-1,-1} - 1, \eta_{12}, \eta_{22} - 1, \eta_{-2,3}, \eta_{-2,-2} - 1, \eta_{-2,-1}, \eta_{13}, \eta_{23} \in I$ and $\eta_{03} \notin \tilde{I} \vee \eta_{-3,3} \notin I$.

By [1, Chapter III, Lemma 4.3] there are $y_1, y_2 \in I$ such that $\eta_{-1,3} + y_2(y_1\eta_{-1,3} + \eta_{-2,3}) \in \text{rad}(R)$. Set $\xi_3 := T_{-1,-2}(y_2)T_{-2,-1}(y_1) \in U$ and $\theta := \xi_3\eta$. Then $\theta \equiv \eta \pmod{I, \tilde{I}, I_0, \tilde{I}_0}$, $\theta_{33} = \eta_{33} \equiv 1 \pmod{\text{rad}(R)}$ and $\theta_{-1,3} = \eta_{-1,3} + y_2(y_1\eta_{-1,3} + \eta_{-2,3}) \in \text{rad}(R)$. Set $f_{21} := T_{3,-1}(-1)$ and $\tau := f_{21}\theta$. Then $\tau_{3*} \equiv e_3^t \pmod{I, I_0}$, $\tau_{33} \equiv 1 \pmod{\text{rad}(R)}$, $\tau_{0,-1} = \theta_{0,-1} + \theta_{03} \equiv \theta_{03} \equiv \eta_{03} \pmod{\tilde{I}}$ and $\tau_{-3,-1} = \theta_{-3,-1} + \theta_{-3,3} \equiv \theta_{-3,3} \equiv \eta_{-3,3} \pmod{I}$. Hence $\tau_{0,-1} \notin \tilde{I} \vee \tau_{-3,-1} \notin I$. Set $\xi_4 := T_{32}(-(\tau_{33})^{-1}\tau_{32})T_{31}(-(\tau_{33})^{-1}\tau_{31})T_{3,-1}(-(\tau_{33})^{-1}\tau_{3,-1})T_{3,-2}(-(\tau_{33})^{-1}\tau_{3,-2}) \in U$ and $\chi := \tau\xi_4$. Then χ has the form

$$\chi = \left(\begin{array}{c|c|c} A''' & t''' & B''' \\ \hline v''' & z''' & w''' \\ \hline C''' & u''' & D''' \end{array} \right) = \left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & \chi_{33} & \chi_{30} & \chi_{3,-3} & 0 & 0 \\ \hline v''' & z''' & w''' \\ \hline C''' & u''' & D''' \end{array} \right)$$

where A''' has the same size as A'' , B''' has the same size as B'' and so on. Further $\chi_{33} \equiv 1 \pmod{\text{rad}(R) \cap I}$, $\chi_{30} \in I_0$, $\chi_{3,-3} \in I$ and $\chi_{0,-1} \notin \tilde{I} \vee \chi_{-3,-1} \notin I$. One can proceed now as in case 1.2.1 or case 1.2.2.

case 1.2.6 Assume that $\eta_{01}, \eta_{02}, \eta_{0,-2}, \eta_{0,-1}, \eta_{03} \in \tilde{I}$ and $\eta_{-3,1}, \eta_{-2,1}, \eta_{-1,1}, \eta_{-3,2}, \eta_{-2,2}$, $\eta_{-1,2}, \eta_{1,-2}, \eta_{2,-2}$, $\eta_{-3,-2}, \eta_{1,-1}, \eta_{2,-1}, \eta_{-3,-1}, \eta_{11} - 1, \eta_{21}, \eta_{-1,3}, \eta_{-1,-2}, \eta_{-1,-1} - 1, \eta_{12}, \eta_{22} - 1, \eta_{-2,3}, \eta_{-2,-2} - 1, \eta_{-2,-1}, \eta_{13}, \eta_{23}$, $\eta_{-3,3} \in I$.

It follows that $\eta_{00} \not\equiv 1 \pmod{\tilde{I}_0}$, since $\eta \notin U_7((R, \Delta), (I, \Omega_{max}^I))$. Hence there is an $a \in J(\Delta)$ such that $(\eta_{00} - 1)a \notin \tilde{I}$. Choose a $b \in R$ such that $(a, b) \in \Delta$ and set $f_{21} := T_1(a, \underline{b})$ and $\tau := f_{21}\eta$. Then $\tau_{31}, \tau_{32}, \tau_{3,-2} = 0$, $\tau_{33} \equiv 1 \pmod{\text{rad}(R) \cap I}$, $\tau_{30} \in I_0$, $\tau_{3,-3}, \tau_{3,-1} \in I$ and $\tau_{0,-1} \notin \tilde{I}$. Set $\xi_3 := T_{3,-1}(-(\tau_{33})^{-1}\tau_{3,-1}) \in U$ and $\chi := \tau\xi_3$. Then χ has the form

$$\chi = \left(\begin{array}{c|c|c} A''' & t''' & B''' \\ \hline v''' & z''' & w''' \\ \hline C''' & u''' & D''' \end{array} \right) = \left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & \chi_{33} & \chi_{30} & \chi_{3,-3} & 0 \\ \hline v''' & z''' & w''' & & & \\ \hline C''' & u''' & D''' & & & \end{array} \right)$$

where A''' has the same size as A'' , B''' has the same size as B'' and so on. Further $\chi_{33} \equiv 1 \pmod{\text{rad}(R) \cap I}$, $\chi_{30} \in I_0$, $\chi_{3,-3} \in I$ and $\chi_{0,-1} \notin \tilde{I}$. One can proceed now as in case 1.2.1 or case 1.2.2.

case 1.3 Assume that $\sigma_{01}, \sigma_{02} \in \tilde{I}$, $\sigma_{-3,1}, \sigma_{-2,1}, \sigma_{-1,1}, \sigma_{-3,2}, \sigma_{-2,2}, \sigma_{-1,2}, \sigma_{3,-3}, \sigma_{3,-2} \in I$ and $\sigma_{3,-1} \notin I$. Set $f_{11} := T_{21}(1)$ and $\rho := f_{11}\sigma$. Clearly $\rho_{01}, \rho_{02} \in \tilde{I}$ and $\rho_{-3,1}, \rho_{-2,1}, \rho_{-1,1}, \rho_{-3,2}, \rho_{-2,2}, \rho_{-1,2} \in I$. Further

$$\rho_{3*} = \begin{pmatrix} 0 & 0 & 1 & 0 & \sigma_{3,-3} & \sigma_{3,-2} + \sigma_{3,-1} & \sigma_{3,-1} \end{pmatrix}.$$

Since $\sigma_{3,-2} \in I$ and $\sigma_{3,-1} \notin I$, $\rho_{3,-2} = \sigma_{3,-2} + \sigma_{3,-1} \notin I$. One can proceed now as in case 1.2.

case 1.4 Assume that $\sigma_{01}, \sigma_{02} \in \tilde{I}$, $\sigma_{-3,1}, \sigma_{-2,1}, \sigma_{-1,1}, \sigma_{-3,2}, \sigma_{-2,2}, \sigma_{-1,2}, \sigma_{3,-3}, \sigma_{3,-2}, \sigma_{3,-1} \in I$. One can proceed as in case 1.2 (σ has the same properties as ζ in case 1.2).

case 2 Assume that (2) in Lemma 42 holds.

Replace h by $[h, T_k(y, z)]$ in case 1.

Part III Assume that $h \in CU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$.

Since $h \notin CU_{2n+1}((R, \Delta), (I, \Omega))$, there is a $g_0 \in EU_{2n+1}(R, \Delta)$ such that $\sigma := [h, g_0] \notin U_{2n+1}((R, \Delta), (I, \Omega))$. Since $h \in CU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$, $\sigma \in U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. By Lemma 39, there is an $f_{11} \in TEU_{2n+1}(R, \Delta)$ such that $(f_{11}\sigma)_{11}$ is left invertible. Set $\kappa := f_{11}\sigma$. Clearly $\kappa_{11} = (f_{11}\sigma)_{11}$ and hence κ_{11} is left invertible. Further $\kappa \in U_{2n+1}((R, \Delta), (I, \Omega_{max}^I)) \setminus U_{2n+1}((R, \Delta), (I, \Omega))$ (note that $U_{2n+1}((R, \Delta), (I, \Omega))$ is E-normal by Remark 31(b) and Lemma 32). Let x be a left inverse of κ_{11} and set $\xi_0 := T_{21}((1 - \kappa_{11} - \kappa_{21})x) \in EU_{2n+1}(I, \Omega) \subseteq H$, $f_{12} := T_{12}(1)$ and $\omega := f_{12}(\xi_0\kappa)$. One checks easily that $\omega_{11} = 1$ and $\omega \in U_{2n+1}((R, \Delta), (I, \Omega_{max}^I)) \setminus U_{2n+1}((R, \Delta), (I, \Omega))$.

case 1 Assume that $q(\omega_{*1}), q(\omega_{*,-1}) \in \Omega$.

Set

$$\xi_2 := T_1(-q^{-1}(\omega_{*,-1})) \prod_{\substack{i=2 \\ i \neq 0}}^{-2} T_{i,-1}(-\omega_{i,-1}) T_{-1}(-q(\omega_{*1})) \prod_{\substack{i=-2 \\ i \neq 0}}^2 T_{i1}(-\omega_{i1}) \in EU_{2n+1}(I, \Omega) \subseteq H$$

where $q^{-1}(\omega_{*,-1}) = (q_1(\omega_{*,-1}), \underline{q_2(\omega_{*,-1})}) \in \Omega^{-1}$. Further set $\tau := \xi_2\omega$. Then there is an $A \in M_{2n-1}(R)$ such that

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly $\tau \in U_{2n+1}((R, \Delta), (I, \Omega_{max}^I)) \setminus U_{2n+1}((R, \Delta), (I, \Omega))$. Hence, by Lemma 28, there is a $j \in \{2, \dots, -2\} \setminus \{0\}$ such that $q(\tau_{*j}) \notin \Omega$ or there is a $y \in J(\Delta)$ such that $(q(\tau_{*0}) - (1, 0)) \bullet y \notin \Omega$.

case 1.1 Assume that there is a $j \in \{2, \dots, n\}$ such that $q(\tau_{*j}) \notin \Omega$.

Set $g_1 := T_{j1}(1)$ and $\zeta := [\tau, g_1]$. It is easy to show that

$$\zeta = T_{-1}(q(\tau_{*j}) \dot{-} (0, -\tau_{-j,j} + \bar{\tau}_{-j,j}\lambda)) \prod_{\substack{i=2 \\ i \neq 0}}^{-2} T_{i1}(\tau_{ij} - \delta_{ij}).$$

Set $\xi_3 := (\prod_{\substack{i=2 \\ i \neq 0}}^{-2} T_{i1}(\tau_{ij} - \delta_{ij}))^{-1} \in EU_{2n+1}(I, \Omega) \subseteq H$ and $g_2 := T_{-1}(q(\tau_{*j}) \dot{-} (0, -\tau_{-j,j} + \bar{\tau}_{-j,j}\lambda))$. Note that g_2 is not (I, Ω) -elementary since $q(\tau_{*j}) \notin \Omega$. Clearly

$$[\xi_2^{f_{12}}(\xi_0^{f_{11}}[h, g_0]), g_1]\xi_3 = \zeta\xi_3 = g_2.$$

Hence $g_2 \in H$, since H is E-normal.

case 1.2 Assume that there is a $j \in \{-n, \dots, -2\}$ such that $q(\tau_{*j}) \notin \Omega$. See case 1.1.

case 1.3 Assume there is a $y \in J(\Delta)$ such that $(q(\tau_{*0}) \dot{-} (1, 0)) \bullet y \notin \Omega$.

Choose a $z \in R$ such that $(y, z) \in \Delta$ and set $g_1 := T_{-1}(y, z)$ and $\zeta := [\tau, g_1]$. It is easy to show that

$$\zeta = T_{-1}((q(\tau_{*0}) \dot{-} (1, 0)) \bullet y) \prod_{\substack{i=2 \\ i \neq 0}}^{-2} T_{i1}(\tau_{i0}y).$$

Set $\xi_3 := (\prod_{\substack{i=2 \\ i \neq 0}}^{-2} T_{i1}(\tau_{i0}y))^{-1} \in EU_{2n+1}(I, \Omega) \subseteq H$ and $g_2 := T_{-1}((q(\tau_{*0}) \dot{-} (1, 0)) \bullet y)$. Note that g_2 is not (I, Ω) -elementary since $(q(\tau_{*0}) \dot{-} (1, 0)) \bullet y \notin \Omega$. Clearly

$$[\xi_2^{f_{12}}(\xi_0^{f_{11}}[h, g_0]), g_1]\xi_3 = \zeta\xi_3 = g_2.$$

Hence $g_2 \in H$, since H is E-normal.

case 2 Assume that $q(\omega_{*1}) \notin \Omega$.

Set $\xi_2 := T_{13}(-\omega_{13})T_{1,-2}(-\omega_{1,-2}) \in EU_{2n+1}(I, \Omega) \subseteq H$ and $\eta := \omega\xi_2$. Then clearly $\eta_{11} = 1$, $\eta_{13} = \eta_{1,-2} = 0$, $\eta \in U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ and $q(\eta_{*1}) \notin \Omega$.

case 2.1 Assume that $q(\eta_{*, -2}) \in \Omega$.

Set $\xi_3 := \prod_{\substack{i=-2 \\ i \neq 0}}^2 T_{i1}(-\eta_{i1}) \in EU_{2n+1}(I, \Omega) \subseteq H$ and $\tau := \xi_3\eta$. Further set $\chi := T_{-1}(\dot{-}q(\eta_{*1})) \in EU_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Then $\zeta := \chi\tau$ has the form

$$\zeta = \begin{pmatrix} 1 & * & * \\ 0 & A & * \\ 0 & 0 & 1 \end{pmatrix}$$

where $A \in M_{2n-1}(R)$. One checks easily that $\zeta \in U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$ and $q(\zeta_{*, -2}) \in \Omega$. Set $g_1 := T_{12}(-1)$. Using the equality $[\alpha\beta, \gamma] = \alpha[\beta, \gamma][\alpha, \gamma]$ one gets that $[\tau, g_1] = [\chi^{-1}\zeta, g_1] = \chi^{-1}[\zeta, g_1][\chi^{-1}, g_1]$. It is easy to show that $[\zeta, g_1] \in EU_{2n+1}(I, \Omega)$ and hence $\chi^{-1}[\zeta, g_1] \in EU_{2n+1}((R, \Delta), (I, \Omega)) \subseteq H$. On the other hand $[\chi^{-1}, g_1] = T_{-1,2}(-q_2(\eta_{*1}))T_{-2}(\dot{-}q(\eta_{*1}))$ by (S1), (SE1) and (SE2). Set $\xi_4 := (\chi^{-1}[\zeta, g_1]T_{-1,2}(q_2(\eta_{*1})))^{-1} \in EU_{2n+1}((R, \Delta), (I, \Omega)) \subseteq H$ and $g_2 := T_{-2}(\dot{-}q(\eta_{*1}))$. Note that g_2 is not (I, Ω) -elementary since $q(\eta_{*1}) \notin \Omega$. Clearly

$$\xi_4[\xi_3^{f_{12}}(\xi_0^{f_{11}}[h, g_0])\xi_2, g_1] = \xi_4[\tau, g_1] = g_2.$$

Hence $g_2 \in H$, since H is E-normal.

case 2.2 Assume that $q(\eta_{*, -2}) \notin \Omega$.

Set $g_1 := T_{-2, -3}(1)$ and $\zeta := [\eta, g_1]$. Then clearly $\zeta_{11} = 1$ and $\zeta \in U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$. Further, by Lemma 41, $q(\zeta_{*, -3}) \equiv q(\eta_{*, -2}) \pmod{\Omega}$ (which implies $\zeta \notin U_{2n+1}((R, \Delta), (I, \Omega))$ since $q(\eta_{*, -2}) \notin \Omega$) and $q(\zeta_{*1}), q(\zeta_{*, -1}) \in \Omega$. One can proceed now as in case 1.

case 3 Assume that $q(\omega_{*1}) \in \Omega$ and $q(\omega_{*, -1}) \notin \Omega$.

See case 2. □

Theorem 44. *Let H be a subgroup of $U_{2n+1}(R, \Delta)$. Then H is E-normal if and only if there is an odd form ideal (I, Ω) of (R, Δ) such that*

$$EU_{2n+1}((R, \Delta), (I, \Omega)) \subseteq H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega)).$$

Further (I, Ω) is uniquely determined, namely it is the level of H .

Proof.

\Rightarrow :

Suppose that H is E-normal. Let (I, Ω) be the level of H . Then

$$EU_{2n+1}((R, \Delta), (I, \Omega)) \subseteq H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega)).$$

by Lemma 43. Let (I', Ω') be an odd form ideal such that

$$EU_{2n+1}((R, \Delta), (I', \Omega')) \subseteq H \subseteq CU_{2n+1}((R, \Delta), (I', \Omega')).$$

Then, by the standard commutator formulas,

$$\begin{aligned} & EU_{2n+1}((R, \Delta), (I, \Omega)) \\ &= [EU_{2n+1}((R, \Delta), (I, \Omega)), EU_{2n+1}(R, \Delta)] \\ &\subseteq [CU_{2n+1}((R, \Delta), (I', \Omega')), EU_{2n+1}(R, \Delta)] \\ &= EU_{2n+1}((R, \Delta), (I', \Omega')) \end{aligned}$$

It is easy to deduce that $I \subseteq I'$ and $\Omega \subseteq \Omega'$. By symmetry it follows that $I = I'$ and $\Omega = \Omega'$.

\Leftarrow :

Clearly

$$\begin{aligned} & [H, EU_{2n+1}(R, \Delta)] \\ &\subseteq [CU_{2n+1}((R, \Delta), (I, \Omega)), EU_{2n+1}(R, \Delta)] \\ &= EU_{2n+1}((R, \Delta), (I, \Omega)) \\ &\subseteq H \end{aligned}$$

and hence H is E-normal. □

4.2. Noetherian case. In this subsection we assume that R is Noetherian. Further we assume that there is a subring C of $\text{Center}(R)$ such that

- (1) $\bar{c} = c$ for any $c \in C$,
- (2) if (I, Ω) is an odd form ideal of (R, Δ) , $(0, x) \in \Omega$ and $c \in C$, then $(0, cx) \in \Omega$ and
- (3) $R_{\mathfrak{m}}$ is semilocal for any maximal ideal \mathfrak{m} of C .

In (3), $R_{\mathfrak{m}}$ denotes the ring $S_{\mathfrak{m}}^{-1}R$ where $S_{\mathfrak{m}} = C \setminus \mathfrak{m}$. If \mathfrak{m} is a maximal ideal of C , set $\overline{(\frac{x}{s})} := \frac{\bar{x}}{\bar{s}}$, $\lambda_{\mathfrak{m}} := \frac{\lambda}{1}$, $\mu_{\mathfrak{m}} := \frac{\mu}{1}$ and $\Delta_{\mathfrak{m}} := \{(\frac{x}{s}, \frac{y}{s^2}) | (x, y) \in \Delta, s \in S_{\mathfrak{m}}\}$. Then $((R_{\mathfrak{m}}, \bar{\cdot}, \lambda_{\mathfrak{m}}, \mu_{\mathfrak{m}}), \Delta_{\mathfrak{m}})$ is an odd form ring. If (I, Ω) is an odd form ideal of (R, Δ) , set $I_{\mathfrak{m}} := \{\frac{x}{s} | x \in I, s \in S_{\mathfrak{m}}\}$ and $\Omega_{\mathfrak{m}} := \{(\frac{x}{s}, \frac{y}{s^2}) | (x, y) \in \Omega, s \in S_{\mathfrak{m}}\}$. Then $(I_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ is an odd form ideal of $(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$. Denote the localisation homomorphism $R \rightarrow R_{\mathfrak{m}}$ by $f_{\mathfrak{m}}$. Clearly $f_{\mathfrak{m}}$ induces a group homomorphism $F_{\mathfrak{m}} : U_{2n+1}(R, \Delta) \rightarrow U_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$.

Lemma 45. *Let (I, Ω) be an odd form ideal of (R, Δ) and $h \in U_{2n+1}(R, \Delta) \setminus CU_{2n+1}((R, \Delta), (I, \Omega))$. Then there is a maximal ideal \mathfrak{m} of C such that $F_{\mathfrak{m}}(h) \in U_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}}) \setminus CU_{2n+1}((R_{\mathfrak{m}}, \Delta_{\mathfrak{m}}), (I_{\mathfrak{m}}, \Omega_{\mathfrak{m}}))$.*

Proof. Since $h \in U_{2n+1}(R, \Delta) \setminus CU_{2n+1}((R, \Delta), (I, \Omega))$, there is a $g \in EU_{2n+1}(R, \Delta)$ such that $\sigma := [h, g] \notin U_{2n+1}((R, \Delta), (I, \Omega))$. We will show that there is a maximal ideal \mathfrak{m} of C such that $F_{\mathfrak{m}}(\sigma) = [F_{\mathfrak{m}}(h), F_{\mathfrak{m}}(g)] \notin U_{2n+1}((R_{\mathfrak{m}}, \Delta_{\mathfrak{m}}), (I_{\mathfrak{m}}, \Omega_{\mathfrak{m}}))$ which implies that $F_{\mathfrak{m}}(h) \notin CU_{2n+1}((R_{\mathfrak{m}}, \Delta_{\mathfrak{m}}), (I_{\mathfrak{m}}, \Omega_{\mathfrak{m}}))$.

case 1 Assume that $\sigma \not\equiv e \pmod{I, \tilde{I}, I_0, \tilde{I}_0}$.

Then either $\sigma_{hb} \not\equiv e_{hb} \pmod{I}$ or $\sigma_{0*} \not\equiv e_0^t \pmod{\tilde{I}, \tilde{I}_0}$ (compare Remark 29).

case 1.1 Assume that $\sigma_{hb} \not\equiv e_{hb} \pmod{I}$.

Then there are $i, j \in \Theta_{hb}$ such that $\sigma_{ij} \not\equiv \delta_{ij} \pmod{I}$. Set $Y := \{c \in C \mid c(\sigma_{ij} - \delta_{ij}) \in I\}$. Since $\sigma_{ij} - \delta_{ij} \notin I$, Y is a proper ideal of C . Hence it is contained in a maximal ideal \mathfrak{m} of C . Clearly $Y \subseteq \mathfrak{m}$ and hence $S_{\mathfrak{m}} \cap Y = \emptyset$. Assume $(F_{\mathfrak{m}}(\sigma))_{ij} - \delta_{ij} = \frac{\sigma_{ij} - \delta_{ij}}{1} \in I_{\mathfrak{m}}$. Then

$$\begin{aligned} \exists x \in I, s \in S_{\mathfrak{m}} : \frac{\sigma_{ij} - \delta_{ij}}{1} &= \frac{x}{s} \\ \Rightarrow \exists x \in I, s, t \in S_{\mathfrak{m}} : t(s(\sigma_{ij} - \delta_{ij}) - x) &= 0 \\ \Rightarrow \exists x \in I, s, t \in S_{\mathfrak{m}} : ts(\sigma_{ij} - \delta_{ij}) &= tx \in I \\ \Rightarrow \exists u \in S_{\mathfrak{m}} : u(\sigma_{ij} - \delta_{ij}) &\in I \\ \Rightarrow \exists u \in S_{\mathfrak{m}} : u &\in Y. \end{aligned}$$

But this contradicts $S_{\mathfrak{m}} \cap Y = \emptyset$. Hence $(F_{\mathfrak{m}}(\sigma))_{ij} - \delta_{ij} \notin I_{\mathfrak{m}}$ and therefore $F_{\mathfrak{m}}(\sigma) \notin U_{2n+1}((R_{\mathfrak{m}}, \Delta_{\mathfrak{m}}), (I_{\mathfrak{m}}, \Omega_{\mathfrak{m}}))$.

case 1.2 Assume that $\sigma_{0*} \not\equiv e_0^t \pmod{\tilde{I}, \tilde{I}_0}$.

One can show that it follows that $(F_{\mathfrak{m}}(\sigma))_{0*} \not\equiv e_0^t \pmod{\tilde{I}_{\mathfrak{m}}, (\tilde{I}_{\mathfrak{m}})_0}$ (see case 1.1). Hence $F_{\mathfrak{m}}(\sigma) \notin U_{2n+1}((R_{\mathfrak{m}}, \Delta_{\mathfrak{m}}), (I_{\mathfrak{m}}, \Omega_{\mathfrak{m}}))$.

case 2 Assume that $\sigma \equiv e \pmod{I, \tilde{I}, I_0, \tilde{I}_0}$.

It follows that either $\exists j \in \Theta_{hb} : q(\sigma_{*j}) \notin \Omega$ or $\exists x \in J(\Delta) : (q(\sigma_{*0}) - (1, 0)) \bullet x \notin \Omega$ since $\sigma \notin U_{2n+1}((R, \Delta), (I, \Omega))$.

case 2.1 Assume that $\exists j \in \Theta_{hb} : q(\sigma_{*j}) \notin \Omega$.

Set $Y := \{c \in C \mid q(\sigma_{*j}) \bullet c \in \Omega\}$. Since $q(\sigma_{*j}) \notin \Omega$, Y is a proper ideal of C . Hence it is contained in a maximal ideal \mathfrak{m} of C . Clearly $Y \subseteq \mathfrak{m}$ and hence $S_{\mathfrak{m}} \cap Y = \emptyset$. Let $x, y \in R$ such that $q(\sigma_{*j}) = (x, y)$. Assume $q((F_{\mathfrak{m}}(\sigma))_{*j}) = (\frac{x}{1}, \frac{y}{1}) \in \Omega_{\mathfrak{m}}$. Then there are a $(x', y') \in \Omega$ and an $s \in S_{\mathfrak{m}}$ such that $(\frac{x}{1}, \frac{y}{1}) = (\frac{x'}{s}, \frac{y'}{s})$. Hence

$$\begin{aligned} \frac{x}{1} &= \frac{x'}{s} \\ \Rightarrow \exists t \in S_{\mathfrak{m}} : t(sx - x') &= 0 \\ \Rightarrow \exists t \in S_{\mathfrak{m}} : tsx &= tx' \end{aligned}$$

and

$$\begin{aligned} \frac{y}{1} &= \frac{y'}{s^2} \\ \Rightarrow \exists u \in S_{\mathfrak{m}} : u(s^2y - y') &= 0 \\ \Rightarrow \exists u \in S_{\mathfrak{m}} : us^2y &= uy'. \end{aligned}$$

It follows that

$$\begin{aligned}
& (tsx, us^2y) = (tx', uy') \\
& \Rightarrow (utsx, t^2u^2s^2y) = (utx, t^2u^2y') \\
& \Rightarrow (x, y) \bullet uts = (x', y') \bullet ut \in \Omega \\
& \Rightarrow q(\sigma_{*j}) \bullet uts \in \Omega \\
& \Rightarrow uts \in Y.
\end{aligned}$$

But this contradicts $S_{\mathfrak{m}} \cap Y = \emptyset$. Hence $q((F_{\mathfrak{m}}(\sigma))_{*j}) \notin \Omega_{\mathfrak{m}}$ and therefore $F_{\mathfrak{m}}(\sigma) \notin U_{2n+1}((R_{\mathfrak{m}}, \Delta_{\mathfrak{m}}), (I_{\mathfrak{m}}, \Omega_{\mathfrak{m}}))$. Thus $\phi_{\mathfrak{m}}(h)$ does not commute with $\phi_{\mathfrak{m}}(g)$.

case 2.2 Assume that $\exists x \in J(\Delta) : (q(\sigma_{*0}) \dot{-} (1, 0)) \bullet x \notin \Omega$.

One can show that it follows that $(q((F_{\mathfrak{m}}(\sigma))_{*0}) \dot{-} (1, 0)) \bullet f_{\mathfrak{m}}(x) \notin \Omega_{\mathfrak{m}}$ (see case 2.1). Hence $F_{\mathfrak{m}}(\sigma) \notin U_{2n+1}((R_{\mathfrak{m}}, \Delta_{\mathfrak{m}}), (I_{\mathfrak{m}}, \Omega_{\mathfrak{m}}))$. \square

Lemma 46. *For any maximal ideal \mathfrak{m} of C there is an $s_0 \in S_{\mathfrak{m}}$ such that $f_{\mathfrak{m}}$ is injective on s_0R . It follows that $F_{\mathfrak{m}}$ is injective on $U_{2n+1}(R, \Delta) \cap GL_{2n+1}(R, s_0R)$, where $GL_{2n+1}(R, s_0R)$ denotes the normal subgroup $\{\sigma \in GL_{2n+1}(R) \mid \sigma \equiv e \pmod{s_0R}\}$ of $GL_{2n+1}(R)$.*

Proof. For any $s \in S_{\mathfrak{m}}$ set $Y(s) := \{x \in R \mid sx = 0\}$. Then for any $s \in S_{\mathfrak{m}}$, $Y(s)$ is an ideal of R . Since R is Noetherian, the set $Z := \{Y(s) \mid s \in S_{\mathfrak{m}}\}$ has a maximal element $Y(s_0)$. Let $s \in S_{\mathfrak{m}}$. Then $Y(s_0), Y(s) \subseteq Y(s_0s)$. Since $Y(s_0)$ is maximal, it follows that $Y(s_0) = Y(s_0s)$ and hence $Y(s) \subseteq Y(s_0)$. It follows that $Y(s_0)$ is the greatest element of Z . Clearly all elements $x \in s_0R$ have the property that $sx = 0$ for some $s \in S_{\mathfrak{m}}$ implies $x = 0$. We will show now that $f_{\mathfrak{m}}$ is injective on s_0R . Assume that $f_{\mathfrak{m}}(x) = f_{\mathfrak{m}}(y)$ where $x, y \in s_0R$. It follows that there is an $s \in S$ such that $s(x - y) = 0$. But this implies $x - y = 0$ since $x - y \in s_0R$. Hence $f_{\mathfrak{m}}$ is injective on s_0R . Clearly the injectivity of $f_{\mathfrak{m}}$ on s_0R implies that $F_{\mathfrak{m}}$ is injective on $U_{2n+1}(R, \Delta) \cap GL_{2n+1}(R, s_0R)$. \square

In the next lemma we will use the following notation. If $t \in C$, set $t\Delta := \Delta \bullet t \dot{+} (0, t\Lambda)$ where $\Lambda = \Lambda(\Delta)$. One checks easily that $(tR, t\Delta)$ is an odd form ideal of (R, Δ) . Now suppose \mathfrak{m} is a maximal ideal of C and $s \in S_{\mathfrak{m}}$. We denote the subset $\{\frac{x}{s} \mid x \in tR\}$ of $(tR)_{\mathfrak{m}}$ by $(1/s)tR$. Further, for any $\epsilon \in \{\pm 1\}$ we denote the subset $\{(\frac{x}{s^2}, \frac{y}{s}) \mid (x, y) \in (t\Delta)^{\epsilon}\}$ of $(t\Delta_{\mathfrak{m}})^{\epsilon}$ by $(1/s)t\Delta^{\epsilon}$ (instead of $(1/s)t\Delta^1$ we sometimes write $(1/s)t\Delta$). For any $N \in \mathbb{N}$ we denote by $E^N((1/s)tR, (1/s)t\Delta)$ the subset of $EU_{2n+1}((tR)_{\mathfrak{m}}, (t\Delta)_{\mathfrak{m}})$ consisting of all products of N elementary matrices of the form $T_{ij}(x)$ where $x \in (1/s)tR$ or $T_i(a)$ where $a \in (1/s)t\Delta^{-\epsilon(i)}$. Instead of $E^N((1/1)tR, (1/1)t\Delta)$ we sometimes write $E^N(tR, t\Delta)$. If $k \in \mathbb{N}$ and M_1, \dots, M_k are subsets of $U_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$, then $M_k(\dots M_2(M_1 E^N((1/s)tR, (1/s)t\Delta)) \dots)$ denotes the set of all product of N matrices of the form $\sigma_1(\dots \sigma_{k-2}(\sigma_k \mu) \dots)$ where $\sigma_i \in M_i \forall i \in \{1, \dots, k\}$ and $\mu \in E^1((1/s)tR, (1/s)t\Delta)$.

Lemma 47. *Let \mathfrak{m} be a maximal ideal of C , $s \in S_{\mathfrak{m}}$ and $t \in C$. Given $K, L, m \in \mathbb{N}$ there are $k, M \in \mathbb{N}$, e.g. $k = (m+2)4^K + 2 \cdot 4^{K-1} + \dots + 2 \cdot 4$ and $M = L \cdot 22^K$, such that*

$$E^K((1/s)tR, (1/s)t\Delta) E^L(s^k tR, s^k t\Delta) \subseteq E^M(s^m tR, s^m t\Delta).$$

Proof. In order to prove the lemma it clearly suffices to prove the case that $L = 1$. We will do this by induction on K .

$K = 1$

Let $m \in \mathbb{N}$. We have to show that

$$E^1((1/s)tR, (1/s)t\Delta) E^1(s^{(m+2)4} tR, s^{(m+2)4} t\Delta) \subseteq E^{22}(s^m tR, s^m t\Delta).$$

Let $\sigma \in E^1((1/s)tR, (1/s)t\Delta)$ and $\tau \in E^1(s^{(m+2)4} tR, s^{(m+2)4} t\Delta)$. We have to show that $\rho := \sigma\tau \in E^{22}(s^m tR, s^m t\Delta)$.

Part I

Assume that σ and τ are short root matrices. Then $\rho \in E^{14}(s^m t R, s^m t \Delta)$ by [5, proof of Lemma 4.1, Case I].

Part II

Assume that σ is an extra short root and τ a short root matrix. Hence there are $h, i, j \in \Theta_{hb}$ where $i \neq \pm j$, $(y, z) \in (1/s)(t\Delta)^{-\epsilon(h)}$ and $x \in (1/1)s^{(m+2)4}tR$ such that $\sigma = T_h(y, z)$ and $\tau = T_{ij}(x)$.

case 1 Assume that $h \neq j, -i$.

Then σ and τ commute by (SE1) in Lemma 23. Hence $\rho = \tau \in E^1(s^m t R, s^m t \Delta)$.

case 2 Assume that $h = j$.

Then

$$\begin{aligned} \rho &= T_j(y, z) T_{ij}(x) T_j(\dot{-}_{-\epsilon(j)}(y, z)) \\ &= T_{ij}(x) \underbrace{T_{ij}(-x) T_j(y, z) T_{ij}(x) T_j(\dot{-}_{-\epsilon(j)}(y, z))}_{(SE2)} \\ &= T_{ij}(x) T_{j,-i}(-z \lambda_{\mathfrak{m}}^{(\epsilon(j)-1)/2} \bar{x} \lambda_{\mathfrak{m}}^{(1-\epsilon(i))/2}) \\ &\quad \cdot T_i(-y \lambda_{\mathfrak{m}}^{(\epsilon(j)-1)/2} \bar{x} \lambda_{\mathfrak{m}}^{(1-\epsilon(i))/2}, x z \lambda_{\mathfrak{m}}^{(\epsilon(j)-1)/2} \bar{x} \lambda_{\mathfrak{m}}^{(1-\epsilon(i))/2}) \\ &\in E^3(s^m t R, s^m t \Delta). \end{aligned}$$

case 3 Assume that $h = -i$.

This case can be reduced to case 2 using (S1).

Part III

Assume that σ and τ are extra short root matrices. Hence there are $h, i \in \Theta_{hb}$, $(x_1, y_1) \in (1/s)(t\Delta)^{-\epsilon(h)}$ and $(x_2, y_2) \in (1/1)(s^{(m+2)4}t\Delta)^{-\epsilon(i)}$ such that $\sigma = T_h(x_1, y_1)$ and $\tau = T_i(x_2, y_2)$.

case 1 Assume that $h \neq \pm i$.

Then

$$\begin{aligned} \rho &= T_h(x_1, y_1) T_i(x_2, y_2) T_h(\dot{-}_{-\epsilon(h)}(x_1, y_1)) \\ &= \underbrace{T_h(x_1, y_1) T_i(x_2, y_2) T_h(\dot{-}_{-\epsilon(h)}(x_1, y_1)) T_i(\dot{-}_{-\epsilon(i)}(x_2, y_2))}_{(E2)} T_i(x_2, y_2) \\ &= T_{h,-i}(-\lambda_{\mathfrak{m}}^{-(1+\epsilon(h))/2} \bar{x}_1 \mu_{\mathfrak{m}} x_2) T_i(x_2, y_2) \\ &\in E^2(s^m t R, s^m t \Delta). \end{aligned}$$

case 2 Assume that $h = i$.

Then

$$\begin{aligned} \rho &= \underbrace{T_i(x_1, y_1) T_i(x_2, y_2) T_i(\dot{-}_{-\epsilon(i)}(x_1, y_1))}_{(E1)} \\ &= T_i(x_2, y_2 - \lambda_{\mathfrak{m}}^{-(\epsilon(i)+1)/2} (\bar{x}_1 \mu_{\mathfrak{m}} x_2 - \bar{x}_2 \mu_{\mathfrak{m}} x_1)) \\ &\in E^1(s^m t R, s^m t \Delta). \end{aligned}$$

case 3 Assume that $h = -i$.

One checks easily that there is an $(a, b) \in (1/1)(s^{(m+2)2}t\Delta)^{-\epsilon(i)}$ such that $(x_2, y_2) = (a, b) \bullet c$ where $c := \frac{s^{m+2}}{1}$. Choose a $p \in \Theta_{hb}$ such that $p \neq \pm h$ and $\epsilon(p) = \epsilon(-h)$. By (SE2),

$$\tau = T_{-h}(x_2, y_2) = T_{ph}(-cb) [T_{-h,p}(c), T_p(a, b)].$$

Hence

$$\begin{aligned}
\rho &= T_h(x_1, y_1) T_{-h}(x_2, y_2) \\
&= T_h(x_1, y_1) (T_{ph}(-cb) [T_{-h,p}(c), T_p(a, b)]) \\
&= \underbrace{T_h(x_1, y_1) T_{ph}(-cb)}_{\text{Part II, case 2}} \underbrace{[T_h(x_1, y_1) T_{-h,p}(c)]}_{\text{Part II, case 3}} \underbrace{T_p(a, b)}_{\text{Part III, case 1}} \\
&= \underbrace{T_{ph}(-cb) T_{h,-p}(y_1 \lambda_m^{(\epsilon(h)-1)/2} \bar{c} \bar{b} \lambda_m^{(1-\epsilon(h))/2})}_{T_1} \times \\
&\quad \times \underbrace{T_p(x_1 \lambda_m^{(\epsilon(h)-1)/2} \bar{c} \bar{b} \lambda_m^{(1-\epsilon(h))/2}, c b y_1 \lambda_m^{(\epsilon(h)-1)/2} \bar{c} \bar{b} \lambda_m^{(1-\epsilon(h))/2})}_{T_2} \times \\
&\quad \times \underbrace{[T_{-p,h}(-c) T_{hp}(c y_1) T_{-p}(c x_1, c^2 y_1)]}_{(SE1)} T_{h,-p}(-\lambda_m^{-(1+\epsilon(h))/2} \bar{x}_1 \mu_m a) T_p(a, b) \\
&= T_1 T_2 [T_{-p}(c x_1, c^2 y_1) T_{-p,h}(-c) T_{hp}(c y_1), T_{h,-p}(-\lambda_m^{-(1+\epsilon(h))/2} \bar{x}_1 \mu_m a) T_p(a, b)] \\
&= T_1 T_2 \underbrace{T_{-p}(c x_1, c^2 y_1)}_{T_3} T_{-p,h}(-c) T_{hp}(c y_1) T_{h,-p}(-\lambda_m^{-(1+\epsilon(h))/2} \bar{x}_1 \mu_m a) T_p(a, b) \times \\
&\quad \times T_{hp}(-c y_1) T_{-p,h}(c) \underbrace{T_{-p}(\bar{\cdot}_{-\epsilon(p)}(c x_1, c^2 y_1)) T_p(\bar{\cdot}_{-\epsilon(p)}(a, b))}_{T_8} \times \\
&\quad \times \underbrace{T_{h,-p}(\lambda_m^{-(1+\epsilon(h))/2} \bar{x}_1 \mu_m a)}_{T_9} \\
&= T_1 T_2 T_3 \underbrace{T_{-p,h}(-c) T_{hp}(c y_1)}_{T_4} \underbrace{T_{-p,h}(-c) T_{h,-p}(-\lambda_m^{-(1+\epsilon(h))/2} \bar{x}_1 \mu_m a)}_{T_5} \times \\
&\quad \times \underbrace{T_{-p,h}(-c) T_p(a, b)}_{T_6} \underbrace{T_{-p,h}(-c) T_{hp}(-c y_1)}_{T_7} T_8 T_9.
\end{aligned}$$

(S5) implies that

$$T_4 = T_{-p}(0, c^2(-y_1 + \lambda_m^{(-1+\epsilon(p))/2} \bar{y}_1 \lambda_m^{(1+\epsilon(p))/2})) T_{hp}(c y_1) \in E^2(s^m t R, s^m t \Delta)$$

and hence

$$T_7 = T_4^{-1} = T_{hp}(-c y_1) T_{-p}(0, -c^2(-y_1 + \lambda_m^{(-1+\epsilon(p))/2} \bar{y}_1 \lambda_m^{(1+\epsilon(p))/2})) \in E^2(s^m t R, s^m t \Delta).$$

Obviously $-\lambda_m^{-(1+\epsilon(h))/2} \bar{x}_1 \mu_m a \in (1/1) s^{2m} t^2 R$. Hence there is a $r \in R$ such that $-\lambda_m^{-(1+\epsilon(h))/2} \bar{x}_1 \mu_m a = f_m(s^{2m} t^2 r)$. Set $d_1 := f_m(s^m t)$, $d_2 := f_m(s^m t r)$ and choose a $k \in \Theta_{hb}$ such that $k \neq \pm h, \pm p$. By (S4),

$$\begin{aligned}
T_5 &= T_{-p,h}(-c) T_{h,-p}(-\lambda_m^{-(1+\epsilon(h))/2} \bar{x}_1 \mu_m a) \\
&= T_{-p,h}(-c) [T_{hk}(d_1), T_{k,-p}(d_2)] \\
&= [T_{-p,h}(-c) T_{hk}(d_1), T_{-p,h}(-c) T_{k,-p}(d_2)] \\
&= [T_{-p,k}(-c d_1) T_{hk}(d_1), T_{kh}(c d_2) T_{k,-p}(d_2)] \\
&\in E^8(s^m t R, s^m t \Delta).
\end{aligned}$$

Further

$$\begin{aligned}
T_6 &= T_{-p,h(-c)} T_p(a, b) \\
&= \underbrace{[T_{-p,h(-c)}, T_p(a, b)]}_{(S1)} T_p(a, b) \\
&= \underbrace{[T_{-h,p}(c), T_p(a, b)]}_{(SE2)} T_p(a, b) \\
&= T_{ph}(cb) T_{-h}(ca, c^2 b) T_p(a, b) \\
&\in E^3(s^m t R, s^m t \Delta).
\end{aligned}$$

Thus

$$\rho = \underbrace{T_1}_2 \underbrace{T_2}_1 \underbrace{T_3}_1 \underbrace{T_4}_2 \underbrace{T_5}_8 \underbrace{T_6}_3 \underbrace{T_7}_2 \underbrace{T_8}_2 \underbrace{T_9}_1 \in E^{22}(s^m t R, s^m t \Delta).$$

Part IV

Assume that σ is a short root and τ an extra short root matrix. All the possibilities which may occur here reduce to one of the cases above.

Thus $\rho \in E^{22}(s^m t R, s^m t \Delta)$.

$K \rightarrow K + 1$

Let $m \in \mathbb{N}$. We have to show that

$$E^{K+1}((1/s)tR, (1/s)t\Delta) E^1(s^k t R, s^k t \Delta) \subseteq E^M(s^m t R, s^m t \Delta).$$

where $k = (m+2)4^{K+1} + 2 \cdot 4^K + \dots + 2 \cdot 4$ and $M = 22^{K+1}$. Set $m' := (m+2)4$ and $M' := 22^K$. Clearly

$$\begin{aligned}
&E^{K+1}((1/s)tR, (1/s)t\Delta) E^1(s^k t R, s^k t \Delta) \\
&\subseteq E^1((1/s)tR, (1/s)t\Delta) (E^K((1/s)tR, (1/s)t\Delta) E^1(s^k t R, s^k t \Delta)) \\
&\stackrel{\text{I.A.}}{\subseteq} E^1((1/s)tR, (1/s)t\Delta) E^{M'}(s^{m'} t R, s^{m'} t \Delta) \\
&\stackrel{K=1}{\subseteq} E^M(s^m t R, s^m t \Delta).
\end{aligned}$$

□

Definition 48. Let G denote a group and A a set of subgroups of G such that

- (1) for any $U, V \in A$ there is a $W \in A$ such that $W \subseteq U \cap V$ and
- (2) for any $g \in G$ and $U \in A$ there is a $V \in A$ such that ${}^g V \subseteq U$.

Then A is called a *base of open subgroups* of $1 \in G$ or just *base for G* .

Lemma 49. Let (I, Ω) be an odd form ideal of (R, Δ) , \mathfrak{m} a maximal ideal of C and $s_0 \in S_{\mathfrak{m}}$. Set $A := \{EU_{2n+1}(ss_0 R, ss_0 \Delta) | s \in S_{\mathfrak{m}}\}$. Then A is a base for $EU_{2n+1}(R, \Delta)$ and $A_{\mathfrak{m}} := \{F_{\mathfrak{m}}(U) | U \in A\}$ is a base for $EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$.

Proof. First we show that A is a base for $EU_{2n+1}(R, \Delta)$.

- (1) Let $U = EU_{2n+1}(ss_0 R, ss_0 \Delta), V = EU_{2n+1}(ts_0 R, ts_0 \Delta) \in A$. Set $W := EU_{2n+1}(sts_0 R, sts_0 \Delta) \in A$. Then clearly $W \subseteq U \cap V$.
- (2) Let $g \in EU_{2n+1}(R, \Delta)$ and $U = EU_{2n+1}(ss_0 R, ss_0 \Delta) \in A$. There is a $K \in \mathbb{N}$ such that g is the product of K elementary matrices. Set $V := EU_{2n+1}((ss_0)^{3 \cdot 4^K + 2 \cdot 4^{K-1} + \dots + 2 \cdot 4} R, (ss_0)^{3 \cdot 4^K + 2 \cdot 4^{K-1} + \dots + 2 \cdot 4} \Delta) \in A$. Then ${}^g V \subseteq U$ by Lemma 47.

Hence A is a base for $EU_{2n+1}(R, \Delta)$. We show now that $A_{\mathfrak{m}}$ is a base for $EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$.

- (1) Let $U = F_{\mathfrak{m}}(EU_{2n+1}(ss_0R, ss_0\Delta))$, $V = F_{\mathfrak{m}}(EU_{2n+1}(ts_0R, ts_0\Delta)) \in A_{\mathfrak{m}}$. Set $W := F_{\mathfrak{m}}(EU_{2n+1}(sts_0R, sts_0\Delta)) \in A_{\mathfrak{m}}$. Then clearly $W \subseteq U \cap V$.
- (2) Let $g \in EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$ and $U = F_{\mathfrak{m}}(EU_{2n+1}(ss_0R, ss_0\Delta)) \in A_{\mathfrak{m}}$. Clearly there are a $K \in \mathbb{N}$ and a $t \in S_{\mathfrak{m}}$ such that $g \in E^K((1/t)R, (1/t)\Delta)$. Set $V := F_{\mathfrak{m}}(EU_{2n+1}((tss_0)^{3 \cdot 4^K + 2 \cdot 4^{K-1} + \dots + 2 \cdot 4}R, (tss_0)^{3 \cdot 4^K + 2 \cdot 4^{K-1} + \dots + 2 \cdot 4}\Delta)) \in A_{\mathfrak{m}}$. Then ${}^gV \subseteq U$ by Lemma 47.

Hence $A_{\mathfrak{m}}$ is a base for $EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$. \square

Lemma 50. *Let (I, Ω) be an odd form ideal of (R, Δ) , \mathfrak{m} a maximal ideal of C and $h' \in U_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}}) \setminus CU_{2n+1}((R_{\mathfrak{m}}, \Delta_{\mathfrak{m}}), (I_{\mathfrak{m}}, \Omega_{\mathfrak{m}}))$. Then given any $U' \in A_{\mathfrak{m}}$, there is a $k \in \mathbb{N}$ and elements $g'_0, \dots, g'_k \in F_{\mathfrak{m}}(EU_{2n+1}(R, \Delta))$, $\epsilon'_0, \dots, \epsilon'_k \in EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$ and $l_1, \dots, l_k \in \{\pm 1\}$ such that g'_k is an elementary matrix in $EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$ which is not $(I_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ -elementary,*

$$\epsilon'_k([\epsilon'_{k-1}(\dots \epsilon'_2([\epsilon'_1([\epsilon'_0 h', g'_0]^{l_1}), g'_1]^{l_2}) \dots), g'_{k-1}]^{l_k}) = g'_k$$

and

$$d'_i g'_i \in U' \quad \forall i \in \{0, \dots, k\}$$

where $d'_i = (\epsilon'_i \dots \epsilon'_0)^{-1}$ ($0 \leq i \leq k$).

Proof. Replace each g_i in the proof of Lemma 43 by an appropriate element of U'_i where $U'_i \in A_{\mathfrak{m}}$ is chosen such that $d'_i U'_i \subseteq U'$ (possible since $A_{\mathfrak{m}}$ is a base for $EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$ by Lemma 49). \square

Lemma 51. *Let (I, Ω) be an odd form ideal of (R, Δ) , \mathfrak{m} a maximal ideal of C and $s_0 \in S_{\mathfrak{m}}$. Set*

$$B := \{EU_{2n+1}(I(xs_0), \Omega(xs_0)) \mid x \in R, \forall s \in S_{\mathfrak{m}} : xs \notin I\} \\ \cup \{EU_{2n+1}(I(a \bullet s_0), \Omega(a \bullet s_0)) \mid a \in \Delta, \forall s \in S_{\mathfrak{m}} : a \bullet s \notin \Omega\}$$

where $(I(xs_0), \Omega(xs_0))$ (resp. $(I(a \bullet s_0), \Omega(a \bullet s_0))$) denotes the odd form ideal defined by xs_0 (resp. $a \bullet s_0$), see Definition 13. Further set $B_{\mathfrak{m}} := F_{\mathfrak{m}}(B)$ and let $A_{\mathfrak{m}}$ be defined as in Lemma 49. Then the following is true.

- (1) If $U' \in A_{\mathfrak{m}}$ and $g' \in EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$ is an elementary matrix which is not $(I_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ -elementary, then

$$V' \subseteq U' g'$$

for some $V' \in B_{\mathfrak{m}}$.

- (2) If $V' \in B_{\mathfrak{m}}$ and $d' \in EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$ is an elementary matrix, then

$$g' \in d' V'$$

for some elementary matrix $g' \in EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$ which is not $(I_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ -elementary.

Proof. Follows from the relations in Lemma 23. \square

Corollary 52. *Let (I, Ω) be an odd form ideal of (R, Δ) , \mathfrak{m} a maximal ideal of C and $s_0 \in S_{\mathfrak{m}}$. If $U' \in A_{\mathfrak{m}}$, $d' \in EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$ and $g' \in EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$ is an elementary matrix which is not $(I_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ -elementary, then*

$$V' \subseteq U' (d' g')$$

for some $V' \in B_{\mathfrak{m}}$.

Proof. If $d' = e$, then we are done, by Lemma 51(1). Assume $d' \neq e$ and write d' as a product $d'_k \dots d'_1$ of nontrivial elementary matrices in $EU_{2n+1}(R_{\mathfrak{m}}, \Delta_{\mathfrak{m}})$. We proceed by induction on k .

case 1 Assume that $k = 1$. Since A_m is a base for $EU_{2n+1}(R_m, \Delta_m)$, there is a $U'_1 \in A_m$ such that $d'_1 U'_1 \subseteq U'$. Clearly

$$\begin{aligned}
& U'(d'_1 g') \\
& \supseteq U'(d'_1 (U'_1 g')) \\
& \stackrel{L.51(1)}{\supseteq} U'(d'_1 V') \text{ (for some } V' \in B_m) \\
& \stackrel{L.51(2)}{\supseteq} U' g'' \text{ (for some elementary matrix } g'' \in EU_{2n}(R_m, \Delta_m) \text{ which is not } (I_m, \Omega_m)\text{-elementary)} \\
& \stackrel{L.51(1)}{\supseteq} V'_1 \text{ (for some } V'_1 \in B_m).
\end{aligned}$$

case 2 Assume that $k > 1$. Set $h' := d'_{k-1} \dots d'_1$. Thus $d' = d'_k \dots d'_1 = d'_k h'$. We can assume by induction on k that given $U'_1 \in A_m$, $U'_1(h' g') \supseteq V'$ for some $V' \in B_m$. Now we proceed similarly to case 1, replacing g' by $h' g'$ and d'_1 by d'_k . Here are the details. Choose $U'_1 \in A_m$ such that $d'_k \phi(U'_1) \subseteq U'$. Clearly

$$\begin{aligned}
& U'(d'_k h' g') \\
& \supseteq U'(d'_k (U'_1(h' g')))) \\
& \stackrel{I.A.}{\supseteq} U'(d'_k V') \\
& \stackrel{L.51(2)}{\supseteq} U' g'' \text{ (for some elementary matrix } g'' \in EU_{2n}(R_m, \Delta_m) \text{ which is not } (I_m, \Omega_m)\text{-elementary)} \\
& \stackrel{L.51(1)}{\supseteq} V'_1 \text{ (for some } V'_1 \in B_m).
\end{aligned}$$

□

Lemma 53. *Let (I, Ω) be an odd form ideal and H an E -normal subgroup of level (I, Ω) . Then $H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega))$.*

Proof. The proof is by contradiction. Suppose $H \not\subseteq CU_{2n+1}((R, \Delta), (I, \Omega))$. Then there is an $h \in H \setminus CU_{2n+1}((R, \Delta), (I, \Omega))$. By Lemma 45 there is a maximal ideal \mathfrak{m} of C such that $h' := F_{\mathfrak{m}}(h) \in U_{2n+1}(R_m, \Delta_m) \setminus CU_{2n+1}((R_m, \Delta_m), (I_m, \Omega_m))$. By Lemma 46 there is an $s_0 \in S_m$ such that $F_{\mathfrak{m}}$ is injective on $K := U_{2n+1}(R, \Delta) \cap GL_{2n+1}(R, s_0 R)$. Let A and A_m be the bases defined in Lemma 49 and choose an $U' \in A_m$. By Lemma 50 there is a $k \in \mathbb{N}$ and elements $g'_0, \dots, g'_k \in F_{\mathfrak{m}}(EU_{2n+1}(R, \Delta))$, $\epsilon'_0, \dots, \epsilon'_k \in EU_{2n+1}(R_m, \Delta_m)$ and $l_1, \dots, l_k \in \{\pm 1\}$ such that g'_k is an elementary matrix in $EU_{2n+1}(R_m, \Delta_m)$ which is not (I_m, Ω_m) -elementary,

$$\epsilon'_k([\epsilon'_{k-1}(\dots \epsilon'_2([\epsilon'_1([\epsilon'_0 h', g'_0]^{l_1}), g'_1]^{l_2}) \dots), g'_{k-1}]^{l_k}) = g'_k \quad (53.1)$$

and

$$d'_i g'_i \in U' \quad \forall i \in \{0, \dots, k\}$$

where $d'_i = (\epsilon'_i \dots \epsilon'_0)^{-1}$ ($0 \leq i \leq k$). By conjugating (53.1) by d'_k we get

$$[\dots [[h', d'_0 g'_0]^{l_1}, d'_1 g'_1]^{l_2} \dots, d'_{k-1} g'_{k-1}]^{l_k} = d'_k g'_k. \quad (53.2)$$

By Corollary 52 there is a $V' \in B_m$ (where B_m is defined as in Lemma 51) such that

$$V' \subseteq U'(d'_k g'_k). \quad (53.3)$$

Let $U \in A$ and $V \in B$ such that $F_{\mathfrak{m}}(U) = U'$ and $F_{\mathfrak{m}}(V) = V'$. Since $d'_i g'_i \in U' \quad \forall i \in \{0, \dots, k\}$, there are $x_0, \dots, x_k \in U$ such that $F_{\mathfrak{m}}(x_i) = d'_i g'_i \quad \forall i \in \{0, \dots, k\}$. Set

$$x := [\dots [[h, x_0]^{l_1}, x_1]^{l_2} \dots, x_{k-1}]^{l_k}.$$

Clearly the l.h.s. of (53.2) equals $F_m(x)$. Further $x \in H$ since H is normalized by $EU_{2n+1}(R, \Delta)$ and $x \in K$ since $x_{k-1} \in U \subseteq K$ and K is normal. It follows that $Ux \subseteq H \cap K$. By (53.2) and (53.3) we have

$$F_m(V) \subseteq F_m(U)F_m(x) = F_m(Ux) \subseteq F_m(H \cap K). \quad (53.4)$$

Since F_m is injective on K , it follows from (53.4) that $V \subseteq H$ (note that $V \subseteq K$). But clearly V contains elementary matrices which are not (I, Ω) -elementary. This contradicts the assumption that H is of level (I, Ω) . Thus $H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega))$. \square

Theorem 54. *Let H be a subgroup of $U_{2n+1}(R, \Delta)$. Then H is E -normal if and only if there is an odd form ideal (I, Ω) of (R, Δ) such that*

$$EU_{2n+1}((R, \Delta), (I, \Omega)) \subseteq H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega)).$$

Further (I, Ω) is uniquely determined, namely it is the level of H .

Proof. See the proof of Theorem 44. \square

4.3. Quasifinite case. In this subsection we assume that R is *quasifinite*. By this we mean that R is the direct limit of subrings R_i ($i \in \Phi$) which are almost commutative (i.e. finitely generated as modules over their centers), involution invariant and contain λ and μ .

Lemma 55. *Each R_i is the direct limit of Noetherian, involution invariant subrings R_{ij} ($j \in \Psi_i$) containing λ and μ such that for any $j \in \Psi_i$ there is a subring C_{ij} of $\text{Center}(R_{ij})$ with the properties*

- (1) $\bar{c} = c$ for any $c \in C_{ij}$,
- (2) if Δ_{ij} is an odd form parameter for R_{ij} , (I_{ij}, Ω_{ij}) an odd form ideal of (R_{ij}, Δ_{ij}) , $(0, x) \in \Omega_{ij}$ and $c \in C_{ij}$, then $(0, cx) \in \Omega_{ij}$ and
- (3) $(R_{ij})_{\mathfrak{m}}$ is semilocal for any maximal ideal \mathfrak{m} of C_{ij} .

Proof. Denote the center of R_i by C_i . Since R_i is almost commutative, there are an $p \in \mathbb{N}$ and $x_1, \dots, x_q \in R_i$ such that $R_i = C_i x_1 + \dots + C_i x_q$. For each $k, l \in \{1, \dots, q\}$ there are $a_1^{(kl)}, \dots, a_q^{(kl)} \in C_i$ such that $x_k x_l = \sum_{p=1}^q a_p^{(kl)} x_p$. Further for each $k \in \{1, \dots, q\}$ there are $b_1^{(k)}, \dots, b_q^{(k)} \in C_i$ such that $\bar{x}_k = \sum_{p=1}^q b_p^{(k)} x_p$.

Finally there are $c_1, \dots, c_q \in C_i$ and $d_1, \dots, d_q \in C_i$ such that $\lambda = \sum_{p=1}^q c_p x_p$ and $\mu = \sum_{p=1}^q d_p x_p$. Set

$$K := \mathbb{Z}[a_p^{(kl)}, \overline{a_p^{(kl)}}, b_p^{(k)}, \overline{b_p^{(k)}}, c_p, \overline{c_p}, d_p, \overline{d_p} | k, l, p \in \{1, \dots, q\}].$$

One checks easily that C_i is a K -algebra and the direct limit of all involution invariant K -subalgebras A_{ij} ($j \in \Psi_i$) of C_i which are finitely generated over K . For any $j \in \Psi_i$ set $R_{ij} := A_{ij} + A_{ij}x_1 + \dots + A_{ij}x_q$. One checks easily that each R_{ij} is an involution invariant subring of R_i containing λ and μ . Further $\varinjlim_j R_{ij} = R_i$. Fix a $j \in \Psi_i$ and let C_{ij} denote the subring of A_{ij} consisting of all finite sums of elements

of the form $a\bar{a}$ and $-a\bar{a}$ where $a \in A_{ij}$. Clearly C_{ij} is a subring of $\text{Center}(R_{ij})$ which has the properties (1) and (2). Property (3) can be shown as in [3, proof of Lemma 8.3]. It remains to show that R_{ij} is Noetherian. Clearly A_{ij} is a finitely generated \mathbb{Z} -algebra and hence also a finitely generated C_{ij} -algebra. Since for any $a \in A_{ij}$

$$a + \bar{a} = (a + 1)(\bar{a} + 1) - a\bar{a} - 1,$$

C_{ij} contains all sums $a + \bar{a}$ where $a \in A_{ij}$. Since any $a \in A_{ij}$ is root of the monic polynomial $X^2 - (a + \bar{a})X + a\bar{a}$, A_{ij} is an integral extension of C_{ij} . Since A_{ij} is an integral extension of C_{ij} and a finitely generated C_{ij} -algebra, A_{ij} is a finitely generated module over C_{ij} by [6, Chapter VII, Proposition 1.2]. Since R_{ij} is finitely generated over A_{ij} , it is a finitely generated C_{ij} -module. Since K is a Noetherian ring, A_{ij} is a Noetherian ring (by Hilbert's Basis Theorem) and hence C_{ij} is a Noetherian ring (by the Eakin-Nagata Theorem). Thus R_{ij} is a Noetherian C_{ij} -module and hence a Noetherian ring. \square

Lemma 56. *Let (I, Ω) be an odd form ideal and H an E-normal subgroup of level (I, Ω) . Then $H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega))$.*

Proof. Let $\sigma \in H$ and $\tau \in EU_{2n+1}(R, \Delta)$. We have to show that $[\sigma, \tau] \in U_{2n+1}((R, \Delta), (I, \Omega))$. Since R is quasifinite, it is the direct limit of almost commutative, involution invariant subrings R_i ($i \in \Phi$) containing λ and μ . By Lemma 55 each R_i is the direct limit of Noetherian, involution invariant subrings R_{ij} ($j \in \Psi_i$) containing λ and μ such that for any $j \in \Psi_i$ there is a subring C_{ij} of $Center(R_{ij})$ with the properties (1)-(3) in Lemma 55. If $i \in \Phi$ and $j \in \Psi_i$, set $\Delta_{ij} := \Delta \cap (R_{ij} \times R_{ij})$. One checks easily that $((R_{ij}, -, \lambda, \mu), \Delta_{ij})$ is an odd form ring. Clearly there are an $i \in \Phi$ and a $j \in \Psi_i$ such that $\sigma \in U_{2n+1}(R_{ij}, \Delta_{ij})$ and $\tau \in EU_{2n+1}(R_{ij}, \Delta_{ij})$. Set $H_{ij} := U_{2n+1}(R_{ij}, \Delta_{ij}) \cap H$. Then H_{ij} is an E-normal subgroup of $U_{2n+1}(R_{ij}, \Delta_{ij})$ and $\sigma \in H_{ij}$. Let (I_{ij}, Ω_{ij}) denote the level of H_{ij} . Then obviously $I_{ij} \subseteq I$ and $\Omega_{ij} \subseteq \Omega$. By Theorem 54,

$$H_{ij} \subseteq CU_{2n+1}((R_{ij}, \Delta_{ij}), (I_{ij}, \Omega_{ij})).$$

Hence $[\sigma, \tau] \in U_{2n+1}((R_{ij}, \Delta_{ij}), (I_{ij}, \Omega_{ij})) \subseteq U_{2n+1}((R, \Delta), (I, \Omega))$. \square

Theorem 57. *Let H be a subgroup of $U_{2n+1}(R, \Delta)$. Then H is E-normal if and only if there is an odd form ideal (I, Ω) of (R, Δ) such that*

$$EU_{2n+1}((R, \Delta), (I, \Omega)) \subseteq H \subseteq CU_{2n+1}((R, \Delta), (I, \Omega)).$$

Further (I, Ω) is uniquely determined, namely it is the level of H .

Proof. See the proof of Theorem 44. \square

5. THE ACTION OF CONJUGATION ON E-NORMAL SUBGROUPS

In this section (R, Δ) denotes an odd form ring and n a natural number. We define for any odd form ideal (I, Ω) and $\sigma \in U_{2n+1}(R, \Delta)$ a relative form parameter ${}^\sigma\Omega$ for I . Further we show that for any involution invariant ideal I the map

$$\begin{aligned} U_{2n+1}(R, \Delta) \times ROFP(I) &\rightarrow ROFP(I) \\ (\sigma, \Omega) &\mapsto {}^\sigma\Omega, \end{aligned}$$

where $ROFP(I)$ denotes the set of all relative odd form parameters for I , is a group action. The main result of this section is Theorem 63, which states that ${}^\sigma U_{2n+1}((R, \Delta), (I, \Omega)) = U_{2n+1}((R, \Delta), (I, {}^\sigma\Omega))$ for any $\sigma \in U_{2n+1}(R, \Delta)$. As a corollary one gets that if $n \geq 3$, R is semilocal or quasifinite and $\sigma \in U_{2n+1}(R, \Delta)$, then ${}^\sigma EU_{2n+1}((R, \Delta), (I, \Omega)) = EU_{2n+1}((R, \Delta), (I, {}^\sigma\Omega))$ and ${}^\sigma CU_{2n+1}((R, \Delta), (I, \Omega)) = CU_{2n+1}((R, \Delta), (I, {}^\sigma\Omega))$. It follows from the sandwich classification of E-normal subgroups in the previous section, that if $n \geq 3$, R is semilocal or quasifinite, $\sigma \in U_{2n+1}(R, \Delta)$ and H is an E-normal of level (I, Ω) , then ${}^\sigma H$ is an E-normal subgroup of level $(I, {}^\sigma\Omega)$.

Definition 58. Let (I, Ω) be an odd form ideal and $\sigma \in U_{2n+1}(R, \Delta)$. Then

$$\begin{aligned} {}^\sigma\Omega &:= \{q(\sigma_{*0}x) \dot{+} (0, y) \mid (x, y) \in \Omega\} \dot{+} \Omega_{min}^I \\ &= \{(q(\sigma_{*0}) \dot{-} (1, 0)) \bullet x \dot{+} (x, y) \mid (x, y) \in \Omega\} \dot{+} \Omega_{min}^I \end{aligned}$$

is called *the relative odd form parameter for I defined by Ω and σ* .

Remark 59.

- (a) One checks easily that ${}^\sigma\Omega$ is a relative odd form parameter for I .
- (b) ${}^\sigma\Omega$ depends not only on σ and Ω but also on I , although this is not expressed in the notation.

Lemma 60. *Let $\sigma \in U_{2n+1}(R, \Delta)$ and $u \in M(I)$. Then*

$$q(\sigma u) \dot{-} q(u) \equiv (q(\sigma_{*0}) \dot{-} (1, 0)) \bullet u_0 \pmod{\Omega_{min}^I}.$$

Proof. Clearly

$$\begin{aligned}
& q(\sigma u) \dot{-} q(u) \\
&= q(\sigma \sum_{i=1}^{-1} e_i u_i) \dot{-} q(u) \\
&= q(\sum_{i=1}^{-1} \sigma_{*i} u_i) \dot{-} q(u) \\
&\stackrel{L.27}{=} (\sum_{-1 \leq i \leq 1} q(\sigma_{*i} u_i)) \dot{+} (0, \sum_{i < j} b(\sigma_{*i} u_i, \sigma_{*j} u_j)) \dot{-} q(u) \pmod{\Omega_{min}^I} \\
&= (\sum_{-1 \leq i \leq 1} q(\sigma_{*i}) \bullet u_i) \dot{+} (0, \sum_{i < j} b(e_i u_i, e_j u_j)) \dot{-} q(u) \\
&= (\sum_{-1 \leq i \leq 1} q(\sigma_{*i}) \bullet u_i) \dot{+} (0, \sum_{i=1}^n \bar{u}_i u_{-i}) \dot{-} q(u) \\
&= (\sum_{-1 \leq i \leq 1} q(\sigma_{*i}) \bullet u_i) \dot{-} (u_0, 0) \\
&= (\sum_{-1 \leq i \leq 1} q(\sigma_{*i}) \bullet u_i) \dot{-} (1, 0) \bullet u_0 \\
&= \sum_{-1 \leq i \leq 1} (q(\sigma_{*i}) \dot{-} (\delta_{i0}, 0)) \bullet u_i \\
&\equiv (q(\sigma_{*0}) \dot{-} (1, 0)) \bullet u_0 \pmod{\Omega_{min}^I}.
\end{aligned}$$

□

Lemma 61. *Let I denote an involution invariant ideal and $ROFP(I)$ the set of all relative odd form parameters for I . Then the map*

$$\begin{aligned}
U_{2n+1}(R, \Delta) \times ROFP(I) &\rightarrow ROFP(I) \\
(\sigma, \Omega) &\mapsto {}^\sigma \Omega
\end{aligned}$$

is a (left) group action.

Proof. In Definition 5 we have defined the Heisenberg quasimodule \mathfrak{H} which equals R^2 as a set. In the same way one can define an R -quasimodule structure on the set $M \times R$ (cf. [8, p. 4753]). The addition is given by

$$\begin{aligned}
\dot{+} : (M \times R) \times (M \times R) &\rightarrow M \times R \\
((u, x), (v, y)) &\mapsto (u, x) \dot{+} (v, y) := (u + v, x + y - b(u, v))
\end{aligned}$$

and the scalar multiplication by

$$\begin{aligned}
\bullet : (M \times R) \times R &\rightarrow M \times R \\
((u, x), a) &\mapsto (u, x) \bullet a := (ua, \bar{a}xa).
\end{aligned}$$

We call this quasimodule the *big Heisenberg quasimodule* and denote it by \mathfrak{H}^* . Set $\Delta^* := \{(u, x) \in \mathfrak{H}^* | q(u) \dot{+} (0, x) \in \Delta\}$, $J(\Delta^*) := \{u \in M | \exists x \in R : (u, x) \in \Delta^*\}$ and $\tilde{I}^* := \{v \in M | b(u, v) \in I \ \forall u \in J(\Delta^*)\}$. Further set

$$(\Omega_{min}^I)^* := \{(0, x - \bar{x}\lambda) | x \in I\} \dot{+} \Delta^* \bullet I$$

and

$$(\Omega_{max}^I)^* := \Delta^* \cap (\tilde{I}^* \times I).$$

We call an R -subquasimodule Ω^* of \mathfrak{H}^* lying between $(\Omega_{min}^I)^*$ and $(\Omega_{max}^I)^*$ a *big relative odd form parameter for I* . Denote the set of all big relative odd form parameters for I by $ROFP^*(I)$. There is a one-to-one correspondence between $ROFP(I)$ and $ROFP^*(I)$ (compare [8, p. 4758]). Namely the maps

$$\begin{aligned} f : ROFP(I) &\rightarrow ROFP^*(I) \\ \Omega &\mapsto f(\Omega) := \{(u, x) | u \in M(I), x \in R, q(u) \dot{+} (0, x) \in \Omega\}. \end{aligned}$$

and

$$\begin{aligned} f^{-1} : ROFP^*(I) &\rightarrow ROFP(I) \\ \Omega^* &\mapsto f^{-1}(\Omega^*) := \{q(u) \dot{+} (0, x) | (u, x) \in \Omega^*\}. \end{aligned}$$

are inverse to each other. Define the map

$$\begin{aligned} \psi : U_{2n+1}(R, \Delta) \times ROFP^*(I) &\rightarrow ROFP^*(I) \\ (\sigma, \Omega^*) &\mapsto \psi((\sigma, \Omega^*)) := {}^\sigma \Omega^*. \end{aligned}$$

where ${}^\sigma \Omega^* = \{(\sigma u, x) | (u, x) \in \Omega^*\}$. Then clearly ψ is a (left) group action. Consider the commutative diagram

$$\begin{array}{ccc} U_{2n+1}(R, \Delta) \times ROFP(I) & \xrightarrow{\phi} & ROFP(I) \\ \downarrow (id, f) & & \uparrow f^{-1} \\ U_{2n+1}(R, \Delta) \times ROFP^*(I) & \xrightarrow{\psi} & ROFP^*(I) \end{array}$$

where $\phi = f^{-1} \circ \psi \circ (id, f)$. Clearly ϕ is a group action since ψ is a group action. Further

$$\phi(\sigma, \Omega) = \{q(\sigma u) \dot{+} (0, x) | u \in M(I), x \in R, q(u) \dot{+} (0, x) \in \Omega\}.$$

It follows from Lemma 60 that $\phi(\sigma, \Omega) = {}^\sigma \Omega$. □

Lemma 62. *Let $\sigma \in U_{2n+1}(R, \Delta)$. Then $\sigma \in \tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$ if and only if one of the equivalent conditions (1)-(4) below holds.*

- (1) $q(\sigma_{*0}x) \equiv q(\sigma'_{*0}x) \equiv (x, 0) \pmod{\Omega}$ for any $x \in J(\Omega)$.
- (2) $(q(\sigma_{*0}) \dot{+} (1, 0)) \bullet x, (q(\sigma'_{*0}) \dot{+} (1, 0)) \bullet x \in \Omega$ for any $x \in J(\Omega)$.
- (3) $(\sigma_{00}x, y + q_2(\sigma_{*0}x)), (\sigma'_{00}x, y + q_2(\sigma'_{*0}x)) \in \Omega$ for any $(x, y) \in \Omega$.
- (4) ${}^\sigma \Omega = \Omega$.

Proof. One checks easily that (1)-(4) are equivalent. We will show now that $\sigma \in \tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$ is equivalent to condition (1).

“ \Rightarrow ”:

Assume that $\sigma \in \tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$. Then (1) holds since $e_0x \in M(I, \Omega)$ for any $x \in J(\Omega)$.

“ \Leftarrow ”:

Assume that (1) holds. Let $u \in M(I, \Omega)$. By Lemma 60, $q(\sigma u) \dot{+} q(u) \equiv (q(\sigma_{*0}) \dot{+} (1, 0)) \bullet u_0 \pmod{\Omega}$ and $q(\sigma^{-1}u) \dot{+} q(u) \equiv (q(\sigma'_{*0}) \dot{+} (1, 0)) \bullet u_0 \pmod{\Omega}$. But (1) implies (2) which implies that $(q(\sigma_{*0}) \dot{+} (1, 0)) \bullet u_0, (q(\sigma'_{*0}) \dot{+} (1, 0)) \bullet u_0 \in \Omega$. Hence $q(\sigma u) \equiv q(\sigma^{-1}u) \equiv q(u) \pmod{\Omega}$ and thus $\sigma \in \tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$. □

Theorem 63. *Let $\sigma \in U_{2n+1}(R, \Delta)$. Then*

$${}^\sigma U_{2n+1}((R, \Delta), (I, \Omega)) = U_{2n+1}((R, \Delta), (I, {}^\sigma \Omega)).$$

In particular, $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$ is the normalizer of $U_{2n+1}((R, \Delta), (I, \Omega))$.

Proof. Assume we have shown that

$${}^\sigma U_{2n+1}((R, \Delta), (I, \Omega)) \subseteq U_{2n+1}((R, \Delta), (I, {}^\sigma \Omega)). \quad (63.1)$$

Since σ and Ω were arbitrarily chosen, it follows that

$$\begin{aligned} \sigma^{-1} U_{2n+1}((R, \Delta), (I, {}^\sigma \Omega)) &\subseteq U_{2n+1}((R, \Delta), (I, \Omega)) \\ \Leftrightarrow U_{2n+1}((R, \Delta), (I, {}^\sigma \Omega)) &\subseteq {}^\sigma U_{2n+1}((R, \Delta), (I, \Omega)). \end{aligned}$$

and hence ${}^\sigma U_{2n+1}((R, \Delta), (I, \Omega)) = U_{2n+1}((R, \Delta), (I, {}^\sigma \Omega))$. Thus it suffices to show (63.1).

Let $\rho \in {}^\sigma U_{2n+1}((R, \Delta), (I, \Omega))$. Then there is a $\tau \in U_{2n+1}((R, \Delta), (I, \Omega))$ such that $\rho = {}^\sigma \tau$. We have to show that $\rho \in U_{2n+1}((R, \Delta), (I, {}^\sigma \Omega))$, i.e.

- (1) $\rho_{hb} \equiv e_{hb} \pmod{I}$ and
- (2) $q(\rho u) \equiv q(u) \pmod{{}^\sigma \Omega} \forall u \in M(R, \Delta)$.

One checks easily that (1) holds. It remains to show that (2) holds. Let $u \in M(R, \Delta)$. Set $\xi := \tau - e$. Clearly $\sigma^{-1}u \in M(R, \Delta)$ since $u \in M(R, \Delta)$. It follows that $\xi\sigma^{-1}u \in M(I, \Omega)$ since $\tau \in U_{2n+1}((R, \Delta), (I, \Omega))$. This implies $\sigma\xi\sigma^{-1}u \in M(I)$. Hence

$$\begin{aligned} &q(\rho u) \\ &= q((e + \sigma\xi\sigma^{-1})u) \\ &= q(u + \sigma\xi\sigma^{-1}u) \\ &\stackrel{L.27}{\equiv} q(u) \dot{+} q(\sigma\xi\sigma^{-1}u) \dot{+} (0, b(u, \sigma\xi\sigma^{-1}u)) \pmod{\Omega_{\min}^I}. \end{aligned}$$

In order to show (2) it suffices to show that $q(\sigma\xi\sigma^{-1}u) \dot{+} (0, b(u, \sigma\xi\sigma^{-1}u)) \in {}^\sigma \Omega$. But

$$\begin{aligned} &q(\sigma\xi\sigma^{-1}u) \dot{+} (0, b(u, \sigma\xi\sigma^{-1}u)) \\ &\equiv (q(\sigma_{*0}) \dot{-} (1, 0)) \bullet (\xi\sigma^{-1}u)_0 \dot{+} q(\xi\sigma^{-1}u) \dot{+} (0, b(u, \sigma\xi\sigma^{-1}u)) \pmod{\Omega_{\min}^I} \end{aligned}$$

by Lemma 60. Hence it suffices to show that $q(\xi\sigma^{-1}u) \dot{+} (0, b(u, \sigma\xi\sigma^{-1}u)) \in \Omega$ in order to show that $q(\sigma\xi\sigma^{-1}u) \dot{+} (0, b(u, \sigma\xi\sigma^{-1}u)) \in {}^\sigma \Omega$. But

$$\begin{aligned} &q(\xi\sigma^{-1}u) \dot{+} (0, b(u, \sigma\xi\sigma^{-1}u)) \\ &= q(\xi\sigma^{-1}u) \dot{+} (0, b(\sigma^{-1}u, \xi\sigma^{-1}u)) \\ &\stackrel{L.27}{\equiv} q(\sigma^{-1}u + \xi\sigma^{-1}u) \dot{-} q(\sigma^{-1}u) \pmod{\Omega_{\min}^I} \\ &= q((e + \xi)\sigma^{-1}u) \dot{-} q(\sigma^{-1}u) \\ &= q(\tau\sigma^{-1}u) \dot{-} q(\sigma^{-1}u) \in \Omega \end{aligned}$$

since $\tau \in U_{2n+1}((R, \Delta), (I, \Omega))$. Thus (2) holds. □

Corollary 64. *If $n \geq 3$ and R is semilocal or quasifinite, then*

$${}^\sigma EU_{2n+1}((R, \Delta), (I, \Omega)) = EU_{2n+1}((R, \Delta), (I, {}^\sigma \Omega))$$

and

$${}^\sigma CU_{2n+1}((R, \Delta), (I, \Omega)) = CU_{2n+1}((R, \Delta), (I, {}^\sigma \Omega))$$

for any $\sigma \in U_{2n+1}(R, \Delta)$. Further $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$ is the normaliser of $EU_{2n+1}((R, \Delta), (I, \Omega))$ and of $CU_{2n+1}((R, \Delta), (I, \Omega))$.

Proof. Follows from Theorem 63 and the standard commutator formulas in Theorem 36. \square

Corollary 65. *Let $n \geq 3$ and R be semilocal or quasifinite. Let $\sigma \in U_{2n+1}(R, \Delta)$ and H be an E -normal of level (I, Ω) . Then ${}^\sigma H$ is an E -normal subgroup of level $(I, {}^\sigma \Omega)$.*

Proof. Follows from Corollary 64 and the sandwich classification of E -normal subgroups given in Section 5. \square

The example below shows that in general (even if R is semisimple and $n \geq 3$)

- (1) $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega)) \neq U_{2n+1}(R, \Delta)$,
- (2) $J(\Omega) \neq J({}^\sigma \Omega)$,
- (3) the group action $(\sigma, \Omega) \mapsto {}^\sigma \Omega$ is not transitive and
- (4) an E -normal subgroup of level (I, Ω) is not normalized by $\tilde{U}_{2n+1}((R, \Delta), (I, \Omega))$.

Example 66. Suppose $R = M_2(\mathbb{F}_2)$, $\bar{x} = x^t \forall x \in R$, $\lambda = 1$, $\mu = 0$ and $\Delta := \Delta_{max} = \{(x, y) \in R \times R | y = y^t\}$. Then R is semisimple and hence semilocal. Let $I = \{0\}$. Then

$$\Omega_{min}^I = \{(0, x - \bar{x}\lambda) | x \in I\} + (\Delta \bullet I) = \{0\} \times \{0\}$$

and

$$\Omega_{max}^I = \Delta \cap (\tilde{I} \times I) = R \times \{0\}.$$

Hence the relative complex form parameters for I correspond to right ideals of R (any relative complex form parameter for I is of the form $J \times \{0\}$ for some right ideal J of R and conversely if J is a right ideal of R , then $J \times \{0\}$ is a relative complex form parameter for I). But there are only 5 right ideals of R (they are in 1-1 correspondence to the subspaces of the \mathbb{F}_2 -vector space $\mathbb{F}_2 \times \mathbb{F}_2$), namely

$$\begin{aligned} J_1 &= \{0\}, \\ J_2 &= \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{F}_2 \right\}, \\ J_3 &= \left\{ \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \mid a, b \in \mathbb{F}_2 \right\}, \\ J_4 &= \left\{ \begin{pmatrix} a & b \\ a & b \end{pmatrix} \mid a, b \in \mathbb{F}_2 \right\} \text{ and} \\ J_5 &= R. \end{aligned}$$

Set $\Omega_i := J_i \times \{0\}$ ($1 \leq i \leq 5$). Then

$$ROFP(I) = \{\Omega_1 = \Omega_{min}^I, \Omega_2, \Omega_3, \Omega_4, \Omega_5 = \Omega_{max}^I\}.$$

It is easy to show that

$$\begin{aligned} orbit(\Omega_1) &= \{\Omega_1\}, \\ orbit(\Omega_2) &= orbit(\Omega_3) = orbit(\Omega_4) = \{\Omega_2, \Omega_3, \Omega_4\} \text{ and} \\ orbit(\Omega_5) &= \{\Omega_5\}. \end{aligned}$$

where for any $i \in \{1, \dots, 5\}$, $orbit(\Omega_i)$ denotes the orbit of Ω_i with respect to the group action

$$\begin{aligned} U_{2n+1}(R, \Delta) \times ROFP(I) &\rightarrow ROFP(I) \\ (\sigma, \Omega) &\mapsto {}^\sigma \Omega. \end{aligned}$$

For example, set

$$\sigma := \begin{pmatrix} e^{n \times n} & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & 0 & e^{n \times n} \end{pmatrix} \in M_{2n+1}(R).$$

By Lemma 20, $\sigma \in U_{2n+1}(R, \Delta)$. Clearly

$$\begin{aligned} & {}^\sigma \Omega_2 \\ &= \{(\sigma_{00}x, q_2(\sigma_{*0}x) + y) \mid (x, y) \in \Omega_2\} \\ &= \{(\sigma_{00}x, 0) \mid x \in J_2\} \\ &= J_3 \times \{0\} \\ &= \Omega_3. \end{aligned}$$

It follows from Lemma 62(4) that $\sigma \notin \tilde{U}_{2n+1}((R, \Delta), (I, \Omega_2))$.

Now set

$$H := \left\{ \begin{pmatrix} e^{n \times n} & 0 & 0 \\ x & \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} & y \\ 0 & 0 & e^{n \times n} \end{pmatrix} \in M_{2n+1}(R) \mid x, y \in M_{1 \times n}(R), a \in \mathbb{F}_2 \right\}.$$

Then H is a subgroup of $U_{2n+1}(R, \Delta)$. Further

$$EU_{2n+1}((R, \Delta), (I, \Omega_{max}^I)) \subseteq H \subseteq U_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$$

and hence H is E -normal of level (I, Ω_{max}^I) by Theorem 44 provided $n \geq 3$. Set

$$\tau := \begin{pmatrix} e^{n \times n} & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & e^{n \times n} \end{pmatrix} \in H.$$

One checks easily that

$${}^\sigma \tau = \begin{pmatrix} e^{n \times n} & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & 0 \\ 0 & 0 & e^{n \times n} \end{pmatrix} \notin H$$

and hence H is not normal in $U_{2n+1}(R, \Delta) = \tilde{U}_{2n+1}((R, \Delta), (I, \Omega_{max}^I))$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRASILIA, BRAZIL

E-mail address: raimund.preusser@gmx.de